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THEORY AND METHODS

STATISTICAL INFERENCE USING PROGRESSIVELY HYBRID CENSORED DATA UNDER EXPONENTIATED EXPONENTIAL DISTRIBUTION WITH BINOMIAL RANDOM REMOVALS

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Summary: In reliability analysis, it is quite common that the failure of any individual or any item may be attributable to more than one cause. Moreover, the observed data are often censored. A hybrid censoring scheme which is a mixture of conventional Type I and Type II censoring schemes is quite useful in life-testing or reliability experiments. Recently Type II progressive censoring schemes have become quite popular for analysing highly reliable data. However, in that case the duration of the experiment can be quite lengthy. Hence, in this paper we introduce a Type II progressively hybrid censoring scheme with random removals, where the number of units removed at each failure time follows a binomial distribution and the experiment terminates at a prespecified time. We derive the likelihood inference and Bayes procedures of the unknown parameters under the assumptions that the lifetime distributions of the different causes are independent and exponentiated exponentially distributed.

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1. Introduction

In life-testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. Saving time on testing and the associated costs are some of the major reasons for censoring. A censoring scheme which can balance between (I) total time spent on the experiment; (II) number of units used in the experiment; and (III) the efficiency of statistical inference based on the results of the experiment, is desirable. The most common censoring schemes are Type I (Time) censoring, where the life-testing experiment will be terminated at a fixed time T; and Type II (Item) censoring, where the life-testing experiment will be terminated as soon as the r-th (r is fixed beforehand) failure is observed. However, the conventional Type I and Type II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. For this reason a more general censoring scheme called progressive Type II right censoring was introduced. It can be briefly described as follows: Consider an experiment in which n units are placed on a life-test. At the time of the first failure, R_1 units are randomly removed from the remaining $n-1$ surviving units. Similarly, at the time of the second failure, R_2 units from the remaining $n-2-R_1$ units are randomly removed. The test continues until the m-th failure at which time, all the remaining $R_m = n-m-R_1-R_2-...-R_{m-1}$ units are removed.

Many authors have discussed the maximum likelihood estimation of unknown parameters of some lifetime distribution under progressive censoring with fixed removal. Papers adopting a random removal scheme are relatively rare. Yuen and Tse (1996) indicated that, for example, the number of patients that drop out of a clinical test at each stage is random and cannot be pre-determined. In some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the pattern of removal at each failure is random. Wu and Chang (2002) and (2003) considered the estimation problem based on exponential and Pareto distributions respectively under a progressive Type II censoring scheme with random removal. In these works, the number of units removed from the test at each failure time is assumed to be random.

The main difference between fixed removals and progressive random removals is that the removals are pre-determined in the former case while they are random in the latter case. Note that m is pre-determined in both cases.

The mixture of Type I and Type II censoring schemes is known as the hybrid censoring scheme. The hybrid censoring scheme was first introduced by Epstein (1954) and (1960), but has recently become quite popular in reliability and life-testing experiments, see for example the work of Chen and Bhattacharya (1988), Childs et al. (2003), Draper and Guttman (1987), Fairbanks, Madasan and Dykstra (1982), Gupta and Kundu (1998) and Jeong, Park and Yum (1996). One of the drawbacks of the conventional Type I, Type II or hybrid censoring schemes is that they do not allow for removal of units at points other than the terminal point of the experiment. One censoring scheme known as Type II progressive censoring scheme, which has this advantage, has becomes very popular in the last few years. It can be described as follows: Consider n units in a study and suppose $m < n$ is fixed before the experiment. Moreover, m other integers, $R_1, ..., R_m$ are also fixed before so that $R_1 + ... + R_m + m = n$. At the time of the first failure, say $X_{1;m;n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure, say $X_{2;m;n}$, R_2 of the remaining units are randomly removed and so on. Finally, at the time of the m-th failure, say $X_{m;m;n}$, the rest of the R_m units are removed. For further details on Type II progressive censoring and for its different advantages, the readers may refer to the recent excellent monograph of Balakrishnan and Aggarwala (2000).

The rest of this paper is organised as follows. Section 2 presents the progressively hybrid Type II censored with binomial removals and the likelihood function. Maximum likelihood and Bayes procedures are discussed in Section 3 and Section 4 respectively. Monte Carlo simulations are presented in Section 5 and finally we conclude the paper in section 6.

2. Model description

The probability density function and of the exponentiated exponential distribution with two parameters λ and θ and cumulative distribution function, introduced by Gupta et al. (1998) and

$$\begin{aligned} f(x;\ \theta,\ \lambda) &= \theta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}, \\ F(x;\ \theta,\ \lambda) &= \left[1-e^{-\lambda x}\right]^{\theta} \end{aligned} \tag{2.1}$$

respectively. By setting $\theta = 1$ the functions given in (2.1) reduce to the probability density and cumulative distribution functions of the exponential distribution, i.e.,

$$f(x;\ \lambda) = \lambda e^{-\lambda x}$$

and

$$F(x;\ \lambda) = 1 - e^{-\lambda x} \tag{2.2}$$

respectively.

Now, suppose n identical items are tested and the lifetime distributions of the n items are denoted by $X_1, ..., X_n$. The integer $m < n$ is fixed at the beginning of the experiment, and $R_1, ..., R_m$ are m fixed integers satisfying

$R_1 + ... + R_m + m = n$. The time point T is also fixed beforehand. At the time of first failure $X_{1;m;n}$, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure $X_{2;m;n}$, R_2 of the remaining units are removed and so on. If the m-th failure $X_{m;m;n}$ occurs before the time point T, the experiment stops at the time point $X_{m;m;n}$. On the other hand suppose the m-th failure does not occur before time point T and only J failures occur before the time point T, where $0 \leq J < m$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T, where $R_J^* = n - (R_1 + ... + R_J - J)$. We denote the two cases as Case I and Case II respectively and call this censoring scheme as the Type II progressively hybrid censoring scheme. Therefore, in presence of Type II progressively hybrid censoring scheme, we have one of the following types of observations:

Case I: $\{X_{1;m;n}, ...X_{m;m;n}\}$ if $X_{m;m;n} < T$ (2.3)

or

Case II: $\{X_{1;m;n}, ...X_{J;m;n}\}$ if $X_{J;m;n} < T < X_{J+1;m;n}$ (2.4)

Note that for Case II, $X_{J;m;n} < T < X_{J+1;m;n} < ... < X_{m;m;n}$ and $X_{J+1;m;n} < ... < X_{m;m;n}$ are not observed see Kundu and Joarder (2006). Suppose that any test unit being dropped out from the life test is independent of the others but with the same removal probability π. Then Tse et al. (2000) indicated that the number of test units removed at each failure time has a binomial distribution.

Under random removal, suppose that r_i is a random variable which is independent of X_i; the joint likelihood function of Type II progressively hybrid censoring scheme will be

Case I: $L(X, R; \theta, \lambda) = L_1(X; \theta, \lambda \setminus R) . P(R \setminus X_{m;m;n} < T)$

where

$$L_1\left(X;\ \theta,\ \lambda \setminus R\right) = \prod_{i=1}^{m} f\left(x_{(i)};\ \theta,\ \lambda\right)\left[1 - F\left(x_{(i)};\ \theta,\ \lambda\right)\right]^{r_i} \qquad (2.5)$$

and $P\left(R \setminus X_{m;m;n} < T\right)$ is the joint probability distribution of removals defined as

$$P\left(R \setminus X_{m;m;n} < T\right) = \frac{(n-m)!}{\prod\limits_{i=1}^{m} r_i!\left(n - m - \sum_{j=1}^{m-1} r_j\right)!} \cdot \pi^{\sum_{j=1}^{m-1} r_j}\left(1-\pi\right)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \qquad (2.6)$$

or

Case II: $L\left(X,\ R;\theta,\lambda\right) = L_2\left(X;\theta,\lambda \backslash R\right).P\left(R \backslash X_{J;m;n} < T < X_{J+1;m;n}\right)$

where

$$L_2\left(X;\theta,\lambda \backslash R\right) = \prod_{i=1}^{J} f\left(x_{(i)};\theta,\lambda\right)\left[1 - F\left(x_{(i)};\theta,\lambda\right)\right]^{r_i}\left[1 - F\left(T;\theta,\lambda\right)\right]^{R_J^*} \qquad (2.7)$$

and $P\left(R \setminus X_{J;m;n} < T < X_{J+1;m;n}\right)$ is the joint probability distribution of removals defined as

$$P\left(R \setminus X_{J;m;n} < T < X_{J+1;m;n}\right) = \frac{(n-J)!}{\prod\limits_{i=1}^{J} r_i!\left(n - J - \sum_{j=1}^{J-1} r_j\right)!}\pi^{\sum_{j=1}^{J-1} r_j}\left(1-\pi\right)^{(J-1)(n-J)-\sum_{j=1}^{J-1}(J-j)r_j} \qquad (2.8)$$

since $P\left(R\right)$ in both cases does not involve the parameters θ and λ.

3. Maximum likelihood estimators

This section discusses the process of obtaining the maximum likelihood estimates of the parameters θ, λ and π based on progressively hybrid censored data with binomial random removals. Both point and interval estimations of the parameters are derived. Based on observations (2.3) and (2.4), the log-likelihood function (without the constant term) can be written as

Case I:

$$\begin{aligned}\log L_1\left(X'\theta,\lambda\setminus R\right) &= m\log\theta + m\log\lambda - \lambda\sum_{i=1}^{m} x_{(i)} \\ &+ (\theta-1)\sum_{i=1}^{m}\log\left(1-e^{-\lambda x_{(i)}}\right) \\ &+ \sum_{i=1}^{m} r_{i\log\left[1-\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\right]}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\log L\left(R\setminus X_{m;m;n}<T\right) &= \sum_{j=1}^{m-1} r_j\log\pi + (m-1)(n-m) \\ &- \sum_{j=1}^{m-1}(m-j)\, r_j\log(1-\pi)\,.\end{aligned}$$

The first partial derivative of $\log L_1\left(X;\theta,\lambda\setminus R\right)$ with respect to $\theta,\ \lambda$ are

$$\begin{aligned}\frac{\partial\log L_1\left(X;\ \theta,\ \lambda\setminus R\right)}{\partial\theta} &= \frac{m}{\theta} + \sum_{i=1}^{m}\log\left(1-e^{-\lambda x_{(i)}}\right) \\ &- \sum_{i=1}^{m} r_i\frac{\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\log\left(1-e^{-\lambda x_{(i)}}\right)}{\left[1-\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\right]} \\ \frac{\partial\log L_1\left(X;\ \theta,\ \lambda\setminus R\right)}{\partial\lambda} &= \frac{m}{\lambda} - \sum_{i=1}^{m} x_{(i)} + (\theta-1)\sum_{i=1}^{m}\frac{x_{(i)}e^{-\lambda x_{(i)}}}{\left(1-e^{-\lambda x_{(i)}}\right)} \\ &-\theta\sum_{i=1}^{m} r_i\frac{x_{(i)}e^{-\lambda x_{(i)}}\left(1-e^{-\lambda x_{(i)}}\right)^{\theta-1}}{\left[1-\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\right]}\end{aligned}$$

Thus, the maximum likelihood estimates $\widehat{\theta}$ and $\widehat{\lambda}$ can be obtained by simultaneously solving the above equations once they have been set equal to zero, that is, by

$$\widehat{\theta} = \frac{m}{\sum_{i=1}^{m} r_i \frac{\left(1-e^{\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}}}{\left[1-\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}}\right]} - \sum_{i=1}^{m} \log\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)} \tag{3.2}$$

$$\widehat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_{(i)} \left[1-\left(\widehat{\theta}-1\right)\frac{e^{-\widehat{\lambda} x_{(i)}}}{\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)} + \widehat{\theta} r_i \frac{e^{-\widehat{\lambda} x_{(i)}}\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\theta-1}}{\left[1-\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}}\right]}\right]}. \tag{3.3}$$

Similarly, the first partial derivative of $\log L\left(R \setminus X_{m;m;n} < T\right)$ with respect to π is

$$\frac{\partial \log L\left(R \setminus X_{m;m;n} < T\right)}{\partial \pi} = \frac{\sum_{j=1}^{m-1} r_j}{\pi} - \frac{(m-1)(n-m) - \sum_{j=1}^{m-1}(m-j) r_j}{(1-\pi)}$$

By setting $\partial \log L\left(R \setminus X_{m;m;n} < T\right)/\partial \pi = 0$, we get the likelihood equation for π. Solving the equation obtained with respect to π, we get the maximum likelihood estimator of π in the following form

$$\widehat{\pi} = \frac{\sum_{j=1}^{m-1} r_j}{(m-1)(n-m) - \sum_{j=1}^{m-1}(m-1-j) r_j} \tag{3.4}$$

Case II:

$$\begin{aligned}\log L_2\left(X;\ \theta,\ \lambda \setminus R\right) &= J\log\theta + J\log\lambda - \lambda\sum_{i=1}^{j} x_{(i)}.\\ &+(\theta-1)\sum_{i=1}^{j}\log\left(1-e^{-\lambda x_{(i)}}\right)\\ &+\sum_{i=1}^{j} r_i\log\left[1-\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\right]\\ &+R_j^*\log\left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}P\left(R \setminus X_{J;m;n} < T < X_{J+1;m;n}\right) &= \sum_{j=1}^{J-1} r_j\log\pi + (J-1)(n-J)\\ &-\sum_{i=1}^{J-1}(J-j)\,r_j\log(1-\pi).\end{aligned}$$

Similarly, the first partial derivative of $\log L_2\left(X;\ \theta,\ \lambda \setminus R\right)$ with respect to $\theta,\ \lambda$ are

$$\begin{aligned}\frac{\partial\log L_2\left(X;\ \theta,\ \lambda \setminus R\right)}{\partial\theta} &= \frac{J}{\theta}+\sum_{i=1}^{J}\log\left(1-e^{-\lambda x_{(i)}}\right)\\ &+\sum_{i=1}^{J} r_i\frac{\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\log\left(1-e^{-\lambda_{(i)}}\right)}{\left[1-\left(1-e^{-\lambda x_{(i)}}\right)^{\theta}\right]}\\ &-R_J^*\frac{\left(1-e^{-\lambda T}\right)^{\theta}\log\left(1-e^{-\lambda T}\right)}{\left[1-(1-e^{-\lambda T})^{\theta}\right]}\end{aligned}$$

$$\frac{\partial \log L_2\left(X;\ \theta,\ \lambda \setminus R\right)}{\partial \lambda} = \frac{J}{\lambda} - \sum_{i=1}^{J} x_{(i)} + (\theta - 1) \sum_{i=1}^{J} \frac{x_{(i) e^{-\lambda x_{(i)}}}}{\left(1 - e^{-\lambda x_{(i)}}\right)}$$
$$-\theta \sum_{i=1}^{J} r_i \frac{x_{(i)} e^{-\lambda x_{(i)}} \left(1 - e^{-\lambda x_{(i)}}\right)^{\theta - 1}}{\left[1 - \left(1 - e^{-\lambda x_{(i)}}\right)^{\theta}\right]}$$
$$-T\theta R_J^* \frac{e^{-\lambda T} \left(1 - e^{-\lambda T}\right)^{\theta - 1}}{\left[1 - \left(1 - e^{-\lambda T}\right)^{\theta}\right]}.$$

Therefore, the maximum likelihood estimates $\widehat{\theta}$ and $\widehat{\lambda}$ can be obtained by simultaneously solving the above equations once they have been set equal to zero,

$$\widehat{\theta} = \frac{J}{-\sum_{i=1}^{J} r_i \frac{\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}} \log\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)}{\left[1-\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}}\right]} + R_J^* \frac{\left(1-e^{-\lambda T}\right)^{\theta} \log\left(1-e^{-\lambda T}\right)}{\left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]} - \sum_{i=1}^{J} \log\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)} \tag{3.6}$$

$$\widehat{\lambda} = \frac{m}{\sum_{i=1}^{m} x_{(i)} \left[1 - \left(\widehat{\theta} - 1\right) \frac{e^{-\widehat{\lambda} x_{(i)}}}{\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)} + \widehat{\theta} r_i \frac{e^{-\widehat{\lambda} x_{(i)}} \left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}-1}}{\left[1-\left(1-e^{-\widehat{\lambda} x_{(i)}}\right)^{\widehat{\theta}}\right]}\right] + T\theta R_J^* \frac{e^{-\lambda T}\left(1-e^{-\lambda T}\right)^{\theta-1}}{\left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]}} \tag{3.7}$$

Similarly, the first partial derivative of $\log L\left(R \setminus X_{J+1;m;n}\right)$ with respect to π is

$$\frac{\partial \log L\left(R \setminus X_{J;m;n}\right) < T < X_{J;1+m;n}}{\partial \pi}$$

$$= \frac{\sum_{j=1}^{J-1} r_j}{\pi} - \frac{(J-1)(n-J) - \sum_{j=1}^{J-1} (J-j)\, r_j}{(1-\pi)}$$

By setting $\partial \log L\left(R \setminus X_{J;m;n} < T < X_{J+1;m;n}\right)/\partial\pi = 0$, we get the likelihood equation for π. Solving the equation obtained with respect to π, we get the maximum likelihood estimator of π in the following form

$$\widehat{\pi} = \frac{\sum_{j=1}^{J-1} r_j}{(J-1)(n-J) - \sum_{j=1}^{J-1}(J-1-j)\, r_j}. \tag{3.8}$$

The Fisher information matrix with random removal will be

$$I\left(\widehat{\lambda},\widehat{\theta},\widehat{\pi}\right) = \begin{bmatrix} I_1\left(\widehat{\lambda},\widehat{\theta}\right) & 0 \\ 0 & I_2\left(\widehat{\pi}\right) \end{bmatrix}$$

where

$$I_1\left(\widehat{\lambda},\widehat{\theta}\right) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = \begin{bmatrix} E\left(\frac{\partial^2 L(\lambda,\theta)}{\partial\theta^2}\right) & E\left(-\frac{\partial^2 L(\lambda,\theta)}{\partial\theta\partial\lambda}\right) \\ E\left(-\frac{\partial^2 L(\lambda,\theta)}{\partial\theta\partial\lambda}\right) & E\left(\frac{\partial^2 L(\lambda,\theta)}{\partial\lambda^2}\right) \end{bmatrix}_{\substack{\lambda=\widehat{\lambda} \\ \theta=\widehat{\theta}}},$$

$$I_2\left(\widehat{\pi}\right) = E\left(-\frac{\partial^2 \ln L(\pi)}{\partial\pi^2}\right)_{\pi=\widehat{\pi}},$$

and

$$-\frac{\partial^2 \ln P(R)}{\partial\pi^2} = \frac{1}{\pi^2}\sum_{j=1}^{D-1} r_j + \frac{1}{(1-\pi)^2}\left[(D-1)(n-D) - \sum_{j=1}^{D-1}(D-j)\, r_j\right].$$

where $D = m$ for Case I and $D = J$ for Case II. The variance-covariance matrix with random removal may be approximated as

$$V\left(\widehat{\lambda},\widehat{\theta},\widehat{\pi}\right) = \begin{bmatrix} V_1\left(\widehat{\lambda},\widehat{\theta}\right) & 0 \\ 0 & V_2\left(\widehat{\pi}\right) \end{bmatrix},$$

where

$$V_1 = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}^{-1}$$

and

$$V_2 = \left[-\frac{\partial^2 \ln P(R)}{\partial\pi^2}\right]^{-1}.$$

It is known that the asymptotic distribution of the maximum likelihood estimators $\widehat{\theta}, \widehat{\lambda}$ and $\widehat{\pi}$, is given by

$$\begin{bmatrix} \widehat{\theta} \\ \widehat{\lambda} \\ \widehat{\pi} \end{bmatrix} \sim N \left[\begin{pmatrix} \widehat{\theta} \\ \widehat{\lambda} \\ \widehat{\pi} \end{pmatrix}, \ V\left(\widehat{\lambda}, \widehat{\theta}, \widehat{\pi},\right) \right]. \tag{3.9}$$

Not that closed form expressions of the expected values of these second order partial derivatives are not readily available. These terms can be evaluated by using numerical methods. Furthermore, define $V_1 = \lim_{n\to\infty} nI_1^{-1}\left(\widehat{\theta}, \widehat{\lambda}\right)$. The joint asymptotic distribution of the maximum likelihood estimators of θ and λ is multivariate normal [see Lawless (1982)]. A numerical technique is needed to obtain the Fisher information matrix and the variance-covariance matrix.

4. Bayes estimators

In this section, we use the Bayes procedure to derive the point and interval estimates of the parameters $\theta, \lambda,$ and π based on progressively hybrid censored data with binomial removals. For this purpose, we need the following additional assumption:

1. The parameters $\theta,\ \lambda$ and π behave as independent random variables.
2. The random variable λ has an exponential distribution with known parameter β as a prior distribution. Namely, the prior probability density function of λ takes the following form
$$g_1(\lambda) = \beta e^{-\beta\lambda} \qquad \beta > 0,\ \beta > 0.$$
3. The random variable θ has the following non-informative type of prior
$$g_2(\theta) = \frac{1}{c} \qquad 0 < \theta\, c.$$
4. π has a beta prior distribution with known parameters $a,\ b$. That is, the prior probability density of function of π is given by
$$g_3(\pi) = \frac{1}{B(a,\ b)} \pi^{a-1} (1-\pi)^{b-1}, \qquad 0 < \pi < 1; \quad a,\ b > 0.$$

5. The joint prior probability density function of $\lambda,\ \theta$ and π is

$$\begin{aligned} g\left(\lambda,\ \theta,\ \pi\right) &= g_1\left(\lambda\right) g_2\left(\theta\right) g_3\left(\pi\right) \\ &= \frac{\beta e^{\beta\lambda}}{cB\left(a,\ b\right)} \pi^{a-1}\left(1-\pi\right)^{b-1}. \end{aligned} \tag{4.1}$$

6. The loss function is

$$l\left[\left(\lambda,\theta,\pi\right),\left(\widehat{\lambda},\widehat{\theta},\widehat{\pi}\right)\right]=\varepsilon_1\left(\lambda-\widehat{\lambda}\right)^2+\varepsilon_2\left(\theta-\widehat{\theta}\right)^2+\varepsilon_3\left(\pi-\widehat{\pi}\right)^2, \quad \varepsilon_1,\varepsilon_2,\varepsilon_3>0.$$

Combining (4.1) with the likelihood function (2.5) and combining (2.6) with the cumulative distribution and probability density functions in (2.1) and using Bayes' theorem, the joint posterior distribution in Case I is derived as follows

$$\omega\left(\lambda,\theta/x,r\right) = \frac{\theta^m\lambda^m e^{-\lambda\left(\beta+\sum_{i=1}^m x_1\right)}\prod_{i=1}^m\left(1-e^{-\lambda x_i}\right)^{\theta-1}\left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i}\pi^{a^*-1}(1-\pi)^{b^*-1}}{j_1}$$

where

$$a^*=a+\sum_{i=1}^{m-1} r_i, \qquad b^*=b+\left(m-1\right)\left(n-m\right)-\sum_{j=1}^{m-1}\left(m-j\right)r_j$$

and

$$\begin{aligned} j_1 = \int_0^c\int_0^1\int_0^\infty &\theta^m\lambda^m e^{-\lambda\left(\beta+\sum_{i=1}^m x_i\right)}\prod_{i=1}^m\left(1-e^{-\lambda x_i}\right)^{\theta-1} \\ &\cdot\left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i}\pi^{a^*-1}\left(1-\pi\right)^{b^*-1}d\lambda\,d\pi\,d\theta. \end{aligned}$$

The marginal posterior of a parameter is obtained by integrating the joint posterior distribution with respect to other parameters and hence the marginal posterior of $\lambda,\ \theta$ and π can be written as

$$\omega\left(\lambda\ /\ x,\ r\right)=\frac{\lambda^m e^{-\lambda\left(\beta+\sum_{i=1}^m x_i\right)}j_2}{j_1} \qquad (0<\lambda<\infty)$$

$$\omega\left(\theta\ /\ x,r\right)=\frac{\theta^m j_3}{j_1}, \qquad (0<\theta<c)$$

and

$$\omega\left(\pi\ /\ x, r\right) = \frac{1}{B\left(a^{*}, b^{*}\right)} \pi^{a^{*}-1} \left(1-\pi\right)^{b^{*}-1}, \qquad (0 < \pi < 1)$$

respectively, where

$$j_2 \int_0^c \int_0^1 \theta^m \sum_{i=1}^m \left(1-e^{-\lambda x_i}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i} \pi^{a^{*}-1} \left(1-\pi\right)^{b^{*}-1}\ dx\ d\theta,$$

$$j_3 \int_0^1 \int_0^\infty \lambda^m e^{-\lambda\left(\beta+\sum_{i=1}^m x_i\right)} \prod_{j=1}^m \left(1-e^{-\lambda x_i}\right)^{\theta-1} \cdot \left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i} \pi^{a^{*}-1} \left(1-\pi\right)^{b^{*}-1}\ d\lambda\ d\pi\ .$$

Note that the posterior distribution of π is beta with parameters a^{*} and b^{*}.

In the same way, combining (4.1) with likelihood function (2.7), (2.8) with cumulative function and probability density function (2.1) and using Bayes' theorem, the joint posterior distribution in Case II is derived as follows

$$\omega\left(\lambda, \theta, \pi/x, r\right) = \frac{\theta^J \lambda^J e^{-\lambda\left(\beta+\sum_{i=1}^J x_i\right)} \prod_{i=1}^J \left(1-e^{-\lambda x_i}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i} \left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]^{r_J^{*}} \pi^{a^{*}-1} \left(1-\pi\right)^{b^{*}-1}}{j_1}$$

where

$$a^{*} = a + \sum_{i=1}^{J-1} r_i, \quad b^{*} = b + \left(J-1\right)\left(n-J\right) - \sum_{i=1}^{J-1} \left(m-j\right) r_j.$$

and

$$j_1 = \int_0^c \int_0^1 \int_0^\infty \theta^J \lambda^J e^{-\lambda\left(\beta+\sum_{i=1}^J x_i\right)} \prod_{i=1}^J \left(1-e^{-\lambda x_i}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i} \left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]^{r_J^{*}} \pi^{a^{*}-1} \left(1-\pi\right)^{b^{*}-1} d\lambda\ \ d\pi\ \ d\theta.$$

The marginal posterior of a parameter is obtained by integrating the joint posterior distribution with respect to other parameters and hence the marginal posterior of $\lambda,\ \theta$ and π can be written as

$$\begin{aligned}\omega\left(\lambda\ /\ x, r\right) &= \frac{\lambda^{J} e^{-\lambda\left(\beta+\sum_{i=1}^{J} x_i\right)} j_2}{j_1}, \\ &= \frac{\theta^{J} j_3}{j_1},\end{aligned}$$

and

$$\omega\left(\pi\ /\ x, r\right) = \frac{1}{B\left(a^{*}, b^{*}\right)} \pi^{a^{*}-1}\left(1-\pi\right)^{b^{*}-1}, \qquad (0 < \pi < 1)$$

respectively, where

$$\begin{aligned} j_2 = \int_0^c \int_0^1 \theta^J \prod_{i=1}^{J} \left(1-e^{-\lambda x_i}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda x_i}\right)^{\theta}\right]^{r_i} \\ \cdot \left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]^{r_J^*} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\ d\pi\ d\theta, \end{aligned}$$

$$\begin{aligned} j_J = \int_0^1 \int_0^\infty \lambda^J e^{-\lambda\left(\beta+\sum_{i=1}^{J} x_i\right)} \prod_{i=1}^{J} \left(1-e^{-\lambda x_i}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda i}\right)^{\theta}\right]^{r_i} \\ \cdot \left[1-\left(1-e^{-\lambda T}\right)^{\theta}\right]^{r_J^*} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\ d\lambda\ d\pi. \end{aligned}$$

Under the squared error loss function the Bayes estimators and its associated minimum posterior risk are the posterior mean and variance, respectively. Therefore, under assumption (6), the Bayes estimate λ, θ and π, say $\widetilde{\lambda}, \widetilde{\theta}$ and $\widetilde{\pi}$, and the associated minimum posterior risk, say $R\left(\widetilde{\lambda}\right)$, $R\left(\widetilde{\theta}\right)$ and $R\left(\widetilde{\pi}\right)$, are given as follows

$$\widetilde{\lambda} = E\left(\lambda\ /\ x\right) = \int_0^\infty \lambda\omega\left(\lambda\ /\ x, r\right) d\lambda = \frac{j_4}{j_1},$$

$$R\left(\widetilde{\lambda}\right) = \int_0^\infty \lambda^2\omega\left(\lambda\ /\ x, r\right) d\lambda - \left(\widetilde{\lambda}\right),$$

$$\widetilde{\theta} = E\left(\theta\ /\ x\right) = \int_0^\infty \theta\omega\left(\theta\omega\left(\theta\ /\ x, r\right) d\theta = \frac{j_5}{j_1}\right),$$

$$R\left(\widetilde{\theta}\right) = \int_0^\infty \theta^2\omega\left(\lambda\ /\ x, r\right) d\theta - \left(\widetilde{\theta}\right),$$

and

$$\widetilde{\pi} = \frac{a^*\left(a^*+1\right)}{\left(a^*+b^*+1\right)\left(a^*+b^*\right)},$$
$$R\left(\widetilde{\pi}\right) = \frac{a^* b^*}{\left(a^*+b^*+1\right)\left(a^*+b^*\right)^2}.$$

where

$$j_4 = \int_0^c \int_0^1 \int_0^\infty \theta^m \lambda^{m+1} e^{-\lambda\left(\beta+\sum_{i=1}^m x_i\right)} \prod_{i=1}^m \left(1-e^{-\lambda x_i}\right)^{\theta-1} \cdot \left[1-\left(1-e^{-\lambda x_i}\right)^\theta\right]^{r_i} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\, d\lambda\, d\pi\, d\theta$$

and

$$j_5 = \int_0^c \int_0^1 \int_0^\infty \theta^{m+1} \lambda^{m} e^{-\lambda\left(\beta+\sum_{i=1}^m x_i\right)} \prod_{i=1}^m \left(1-e^{-\lambda x_i}\right)^{\theta-1} \cdot \left[1-\left(1-e^{-\lambda x_i}\right)^\theta\right]^{r_i} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\, d\lambda\, d\pi\, d\theta$$

for Case I, and

$$j_4 = \int_0^c \int_0^1 \int_0^\infty \theta^J \lambda^{J+1} e^{-\lambda\left(\beta+\sum_{i=1}^J x_i\right)} \prod_{i=1}^J \left(1-e^{-\lambda x_i}\right)^{\theta-1} \cdot \left[1-\left(1-e^{-\lambda x_i}\right)^\theta\right]^{r_i} \left[1-\left(1-e^{-\lambda T}\right)^\theta\right]^{r_J^*} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\, d\lambda\, d\pi\, d\theta$$

and

$$j_5 = \int_0^c \int_0^1 \int_0^\infty \theta^{J+1} \lambda^{J} e^{-\lambda\left(\beta+\sum_{i=1}^J x_i\right)} \prod_{i=1}^J \left(1-e^{-\lambda x_i}\right)^{\theta-1} \cdot \left[1-\left(1-e^{-\lambda x_i}\right)^\theta\right]^{r_i} \left[1-\left(1-e^{-\lambda T}\right)^\theta\right]^{r_J^*} \pi^{a^*-1}\left(1-\pi\right)^{b^*-1}\, d\lambda\, d\pi\, d\theta$$

for Case II.

Note that: Many special cases can be obtained from results derived in Sections 3 and 4;

a. If one uses progressively hybrid censored data with fixed removal and $\theta = 1$, then the exponential distribution results as special case from the results by Kundu and Joarder (2006).

b. When all m failures occurs before time point T and $\theta = 1$ then the results in Sections 3 and 4 reduce to an exponential distribution under progressively hybrid censored data with random removal. These results agree with those established by Sarhan and Abuammoh (2008).

c. Type II censoring is obtained as a special case when $r_i = 0$ for $i = 1, 2, ..., m-1$ and $r_m = n - m$.

d. If $r_i = 0$ for $i = 1, 2, ..., m$ progressively hybrid censored data with random removal results reduces to complete sample.

5. Simulation results

Since the performance of the different methods cannot be compared theoretically, we use Monte Carlo simulations to compare different methods for different parameter values and for different sampling schemes. The use of different sampling schemes means that different sets of R_i's were obtained as binomial random variable and for different T values. All the programs are written in MATHCAD (13).

Before progressing further, we first describe how we generate Type II progressively hybrid censored data with binomial random removals. The following algorithm is followed to obtain these samples.

1. Specify the values of n, m, T.
2. Specify the values of θ, λ and π.
3. Generating a random number r_1 from binomial $(n - m, \pi)$.
4. Generating a random number r_i from binomial $\left(n - m - \sum_{l=1}^{i=1} r_l, \pi\right)$ for each $i, i = 2, 3, ..., m-1$.
5. Set r_m according to the following relation

$$r_m = \left\{ n - m - \sum_{l=1}^{i=1} r_l, \quad \text{if} \quad n - m - \sum_{l=1}^{i=1} r_l > 0. \right\}$$

6. Generate a random sample with size m from an exponentiated exponential (λ, θ) distribution and sort it.
7. For given $n, m, R_1, ..., R_m$ we generate ; If $X_{m;m;n} < T$ then we have Case I and the corresponding sample is $\{(X_{1;m;n}, R_1), ..., (X_{m;m;n}, R_m)\}$. If $X_{m;m;n} > T$, then we have Case II and we find J, such that $X_{J;m;n} < T < X_{J+1;m;n}$. The corresponding Type II progressively hybrid censored data with binomial random removals is $\{(X_{1;m;n}, R_1), ..., (X_{J;m;n}, R_J)\}$ and R_j^*, where R_J^* is defined as before. We consider different $n; m; T$, and the different sampling schemes. Without loss of generality we take $\lambda = 1,\ \theta = 2$ and $\pi = 0.3,\ 0.4$ and 0.5 in each case. We compute the estimators using (I) maximum likelihood estimators (II) Bayes estimators. Note that: There is no closed form solution to the above equations, and iterative numerical search can be used to obtain the MLEs from the above likelihood equations depending on the initial guesses of θ, λ and π which can be chosen as following
$$\widehat{\theta} = \frac{-m}{\sum\limits_{i=1}^{m} \log\left(1 - e^{-\widehat{\lambda} x_{(i)}}\right)},$$
$$\widehat{\lambda} = \frac{m}{\sum\limits_{i=1}^{m} x_{(i)} \left[1 - \left(\widehat{\theta} - 1\right) \frac{e^{-\lambda x_{(i)}}}{\left(1 - e^{-\widehat{\lambda} x_{(i)}}\right)}\right]}$$
and
$$\widehat{\pi} = \frac{\sum\limits_{j=1}^{m-1} r_j}{(m-1)}$$
8. The above steps are repeated 1000 times for each sample size and the values averaged.

Table 1. Average estimates of θ, λ and π are presented, under the maximum likelihood estimation for different size and different sampling schemes

π	(n,m)	$T=0.5$			$T=1.00$			$T=2.00$		
		$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$
π=0.3	(15,5)	2.92	0.97	0.24	2.16	1.20	0.26	2.39	1.43	0.27
	(25,5)	2.63	0.98	0.20	2.07	1.21	0.22	2.11	1.45	0.23
	(50,5)	2.35	0.99	0.26	2.59	1.23	0.28	2.02	1.46	0.29
	(100,5)	2.07	0.98	0.22	2.30	1.24	0.24	2.54	1.48	0.25
	(25,10)	2.78	1.02	0.28	2.22	1.25	0.24	2.25	1.49	0.21
	(50,10)	2.50	1.03	0.24	2.73	1.26	0.26	2.26	1.50	0.25
	(100,10)	2.34	1.01	0.21	2.58	1.21	0.22	2.82	1.48	0.23
	(50,15)	1.78	1.03	0.23	2.81	1.27	0.24	2.29	1.57	0.22
	(100,15)	1.66	1.20	0.26	2.90	1.44	0.27	2.84	1.67	0.23
π=0.4	(15,5)	2.96	0.99	0.36	2.25	1.26	0.42	2.41	1.44	0.46
	(25,5)	2.68	1.07	0.31	2.08	1.30	0.41	2.18	1.48	0.45
	(50,5)	2.36	1.02	0.42	2.68	1.33	0.42	2.03	1.54	0.34
	(100,5)	2.15	1.04	0.41	2.38	1.29	0.39	2.56	1.50	0.42
	(25,10)	2.78	1.03	0.46	2.30	1.26	0.39	2.26	1.49	0.38
	(50,10)	2.54	1.09	0.44	2.75	1.26	0.39	2.32	1.57	0.38
	(100,10)	2.38	1.01	0.34	2.61	1.30	0.39	2.88	1.50	0.40
	(50,15)	1.87	1.07	0.39	2.89	1.34	0.36	2.31	1.63	0.50
	(100,15)	1.67	1.22	0.38	3.00	1.48	0.39	2.85	1.70	0.52
π=0.5	(15,5)	3.00	1.01	0.46	2.35	1.32	0.50	2.42	1.50	0.56
	(25,5)	2.78	1.09	0.41	2.16	1.36	0.57	2.21	1.50	0.52
	(50,5)	2.42	1.10	0.54	2.70	1.38	0.57	2.12	1.62	0.54
	(100,5)	2.24	1.08	0.50	2.46	1.34	0.58	2.64	1.58	0.55
	(25,10)	2.81	1.06	0.59	2.38	1.35	0.56	2.32	1.56	0.49
	(50,10)	2.63	1.16	0.52	2.79	1.33	0.57	2.40	1.63	0.55
	(100.10)	2.43	1.11	0.50	2.69	1.39	0.52	2.91	1.53	0.55
	(50,15)	1.90	1.10	0.50	2.94	1.38	0.49	2.40	1.69	0.46
	(100,15)	1.71	1.32	0.53	3.01	1.49	0.50	2.90	1.74	0.45

Table 2. Average estimates of θ, λ and π are presented, under Byes estimation for different size and different sampling schemes

π	(n,m)	$T=0.5,\ \pi=0.3$			$T=1.00,\ \pi=0.3$			$T=2.00,\ \pi=0.3$		
		$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\pi}$
π=0.3	(15,5)	2.94	1.00	0.29	2.19	1.22	0.29	2.43	1.51	0.33
	(25,5)	2.65	1.01	0.25	2.20	1.24	0.25	2.14	1.52	0.29
	(50,5)	2.38	1.02	0.31	2.62	1.25	0.31	2.46	1.53	0.35
	(100,5)	2.09	1.04	0.27	2.34	1.26	0.27	2.57	1.55	0.31
	(25,10)	2.80	1.05	0.33	2.35	1.28	0.27	2.29	1.56	0.27
	(50,10)	2.53	1.06	0.29	2.76	1.28	0.29	2.69	1.57	0.31
	(100,10)	2.37	1.06	0.26	2.61	1.27	0.25	2.85	1.59	0.29
	(50,15)	1.81	1.06	0.28	2.85	1.29	0.27	2.88	1.59	0.28
	(100,15)	1.69	1.24	0.31	2.93	1.46	0.30	2.87	1.75	0.29
π=0.4	(15,5)	2.98	1.05	0.46	2.23	1.24	0.47	2.56	1.56	0.48
	(25,5)	2.65	1.05	0.36	2.29	1.27	0.35	2.19	1.55	0.45
	(50,5)	2.41	1.08	0.42	2.69	1.34	0.46	2.52	1.60	0.46
	(100,5)	2.12	1.05	0.39	2.42	1.31	0.39	2.62	1.57	0.43
	(25,10)	2.86	1.11	0.49	2.36	1.31	0.37	2.39	1.64	0.37
	(50,10)	2.55	1.15	0.41	2.81	1.36	0.42	2.71	1.65	0.45
	(100,10)	2.41	1.15	0.46	2.65	1.35	0.42	2.90	1.61	0.46
	(50,15)	1.85	1.12	0.47	2.85	1.36	0.46	2.90	1.67	0.43
	(100,15)	1.75	1.30	0.46	2.98	1.53	0.48	2.97	1.78	0.46
π=0.5	(15,5)	3.07	1.13	0.59	2.30	1.27	0.65	2.62	1.57	0.66
	(25,5)	2.74	1.08	0.50	2.35	1.36	0.51	2.23	1.59	0.55
	(50,5)	2.48	1.10	0.56	2.70	1.35	0.63	2.57	1.67	0.59
	(100,5)	2.18	1.09	0.57	2.45	1.40	0.57	2.63	1.61	0.55
	(25,10)	2.86	1.19	0.66	2.39	1.34	0.53	2.46	1.68	0.54
	(50,10)	2.65	1.22	0.56	2.82	1.46	0.53	2.81	1.70	0.63
	(100,10)	2.43	1.21	0.64	2.68	1.40	0.60	2.94	1.66	0.64
	(50,15)	2.88	1.13	0.57	2.91	1.45	0.57	3.00	1.74	0.62
	(100,15)	2.84	1.35	0.61	3.05	1.58	0.63	3.06	1.87	0.66

Note that: The increasing of the removal probability π means more items are removed, so variance – covariance matrix is decreasing.

6. Conclusions

This study compares the effect of the removal probability π from a new censoring scheme, namely the Type II progressively hybrid censoring with binomial random removal assuming that the lifetime distributions are Exponentiated Exponentially distributed. Also, we obtain the maximum likelihood estimators of the unknown parameters. A Bayesian estimate of the unknown parameters is also proposed and it is observed that the Bayes estimate with respect to prior works quite well in this case.

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