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Statistical Analysis of Exponentiated Weibull Family under Type I Progressive Interval Censoring with Random Removals

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Abstract: This paper considers the analysis of exponentiated Weibull family distributed lifetime data observed under Type I progressive interval censoring with random removals, where the number of units removed at each failure time follows a binomial distribution. Maximum likelihood estimators of the parameters and their asymptotic variances are derived. The formula to compute the expected duration is given. An example is discussed to illustrate the application of results under this censoring scheme.

Keywords: The exponentiated Weibull family; Maximum likelihood estimation; interval censoring; Progressive Type I censoring; Random removal; Expected duration.

INTRODUCTION

Progressively Type-I Interval Censoring is a union of Type-I interval censoring and progressive censoring. A Progressively Type-I Interval Censored Sample is collected as follows: n units are put on life test at time $T_0 = 0$. Units are observed at pre-set times T_1, T_2,.....T_m. (m is also fixed). At these times, r_1, r_2,.....r_m live units are removed from experimentation, respectively. The values r_1, r_2,......r_m may be pre-specified as percentages of the remaining live units or, alternatively, r_i units available for removal. In this case, the number of live units removed at time T_i is r_i = min(r_i, number of units remaining), $I = 1,2,.....,m$-1 Again r_m equals all remaining units at time T_m, when experimentation is scheduled to terminate.

Suppose a progressively Type-I interval censored sample is collected as described above, beginning with a random sample of n units with a continuous lifetime distribution $F(x,\theta)$ and let $k_1,k_2,......,k_m$ denote the number of units known to have failed in the intervals, $(0, T_1]$, $(T_1,T_2]$,.....,$(T_{m-1},T_m]$, respectively. Then, based on this observed data, the joint likelihood function will be[1]

$$L(X;\theta\setminus R) = C\prod_{i=1}^{m}\left[F(T_i;\theta) - F(T_{i-1};\theta)\right]^{k_i}\left[1 - F(T_i;\theta)\right]^{r_i} \qquad (1.1)$$

where C is constant.

In many industrial processes, life test is conducted in order to assess the quality of product. Typically, n products are placed under test and their times to failure are observed. These observed lifetimes are then used to estimate the life distribution of product. However, in many applications, life tests are usually terminated before the complete lifetimes of the n products are observed. This results in a censored test. Data from this censored test consist of times to failure on failed units and running times on unfailed units. In these works, the number of units being removed from the test at each failure time is assumed to be fixed. However, in many practical situations, these numbers may occur at random. For example, the number of patients drop out from a clinical test at each stage is random and cannot be predetermined. In some industrial experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed completely. In these cases, the pattern of removal at the each failure is random. there is not any work in the literature which considers the cases that the number of units being removed at each failure time is random. Thus, there is a need to develop models and estimation results to incorporate the cases with random removals at each failure time.

The main difference between progressive interval type I censoring with fixed removal and progressive interval type I censoring with random removals and denote it by PICR is that the removals are pre-determined in the former case while they are random in the latter case. Note that m is pre-determined in both cases. However, many practical applications suggest that it is more flexible to have removals random to accommodate the unexpected dropout of experimental subjects.

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Although progressive censoring occurs frequently in many applications, there are relatively few works on it. Some early works can be found in Cohen[3], Mann[6], Thomas & Wilson[10], Viveros & Balakrishnan[13]. Readers can refer to the book Balakrishnan & Aggarwala[2] for more details on the methods and applications of this topic. However, all these works assumed that the number of units being removed from the test is fixed in advance. In practice, it is impossible to pre-determine the removal pattern. Thus, Yuen & Tse[15] and Yang &Yuen[14] considered the estimation problem when lifetimes collected under a Type II progressive censoring with random removals and Kendell & Anderson[5] point out that the expected duration under grouped data.

Model: The probability density function of the exponentiated Weibull family with two shape parameters β and θ, and scale parameter α given by

$$f(x,\alpha,\beta,\theta) = \left(\frac{\theta.\beta}{\alpha}\right).\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \left[1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\theta-1} \qquad (2.1)$$

where $0 < x < \infty, \quad \alpha, \beta \ and\ \theta \geq 0$; the corresponding cumulative distribution function is[7]

$$F(x;\alpha,\beta,\theta) = \left[1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\theta} \qquad (2.2)$$

From equation (2.1), different special distributions can be obtained such as:

1) For, $\beta = 2$ the probability density function and distribution function for the exponentiated exponential distribution introduced by Gupta *et al.*,[4] will be

$$f(x;\theta,\alpha) = \left(\frac{\theta}{\alpha}\right).e^{-\left(\frac{x}{\alpha}\right)}\left(1-e^{-\left(\frac{x}{\alpha}\right)}\right)^{\theta-1}$$

$$F(x;\alpha,\beta,\theta) = \left[1-e^{-\left(\frac{x}{\alpha}\right)}\right]^{\theta} \qquad (2.3)$$

respectively.

2) For $\beta = 2$, the two parameter Burr type X distribution with probability density function and distribution function are given by

$$f(x,\alpha,\theta) = \left(\frac{2\theta}{\alpha^2}\right).xe^{-\left(\frac{x}{\alpha}\right)^2}\left[1-e^{-\left(\frac{x}{\alpha}\right)^2}\right]^{\theta-1}$$

$$F(x;\alpha,\beta,\theta) = \left[1-e^{-\left(\frac{x}{\alpha}\right)^2}\right]^{\theta} \qquad (2.4)$$

respectively.

3) For $\theta = 1$, the probability density function for the Weibull distribution and cumulative distribution function will be

$$f(x;\alpha,\beta) = \left(\frac{\beta}{\alpha}\right).\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$$

$$F(x;\alpha,\beta) = 1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \qquad (2.5)$$

respectively.

4) By taking $\theta = 1$ and $\beta = 1$, the probability density function for the exponential distribution and cumulative distribution function are given by,

$$f(x;\alpha) = (\frac{1}{\alpha}).e^{-\frac{x}{\alpha}}$$

$$F(x;\alpha) = 1 - e^{-\frac{x}{\alpha}} \qquad (2.6)$$

respectively.

5) The probability density function and distribution function for the Rayleigh distribution may be obtained by putting $\theta = 1$ and $\beta = 2$, that is

$$f(x,\alpha) = (\frac{2}{\alpha}).(\frac{x}{\alpha}).e^{-(\frac{x}{\alpha})^2}$$

$$F(x;\alpha) = 1 - e^{-(\frac{x}{\alpha})^2} \qquad (2.7)$$

Mle with Fixed Removal: Following Aggarwala[1], using (1.1) and (2.2); the likelihood function under progressively type-I interval censored with fixed removal will be

$$L_1(X;\alpha,\beta,\theta \setminus R) = \left\{ \prod_{i=1}^{m} \left[[1-e^{-(T_i/\alpha)^{\beta}}]^{\theta} - [1-e^{-(T_{i-1}/\alpha)^{\beta}}]^{\theta} \right]^{k_i} [1-[1-e^{-(T_i/\alpha)^{\beta}}]^{\theta}]^{r_i} \right\}. \qquad (3.1)$$

The expression (3.1) is derived conditional on r_i; each r_i can be of any integer value between 0 and n-m-(r_1+...+r_{i-1}). The logarithm of the likelihood function (3.1) will be

$$\ln L_1(X;\alpha,\beta,\theta \setminus R) = \sum_{i=1}^{m} k_i \ln\left[F(T_i) - F(T_{i-1})\right] + \sum_{i=1}^{m} r_i \ln\left[1 - F(T_i)\right] \qquad (3.2)$$

Where

$$F(T_i) = [1 - e^{-\left(\frac{T_i}{\alpha}\right)^{\beta}}]^{\theta}$$

Thus, the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ can beobtained by maximizing (3.2) with respect to $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$; that is, by simultaneously solving the estimating equations,

$$\sum_{i=1}^{m} \frac{k_i}{F(T_i) - F(T_{i-1})} (\frac{\partial F(T_i)}{\partial \hat{\alpha}} - \frac{\partial F(T_{i-1})}{\partial \hat{\alpha}}) + \sum_{i=1}^{m} \frac{r_i}{1 - F(T_i)} (\frac{\partial[1 - F(T_i)]}{\partial \hat{\alpha}}) = 0 \qquad (3.3)$$

Where

$$\frac{\partial F(T_i)}{\partial \hat{\alpha}} = (\frac{\hat{\beta}\hat{\theta}}{\hat{\alpha}})(P_i)^{\hat{\theta}-1}(1-P_i)\ln(1-P_i) \quad P_i = 1 - e^{-(T_i/\hat{\alpha})^{\hat{\beta}}}$$

and

$$\frac{\partial[1 - F(T_i)]}{\partial \hat{\alpha}} = -\frac{\partial F(T_i)}{\partial \hat{\alpha}}$$

$$\sum_{i=1}^{m} \frac{k_i}{F(T_i) - F(T_{i-1})} (\frac{\partial F(T_i)}{\partial \hat{\beta}} - \frac{\partial F(T_{i-1})}{\partial \hat{\beta}}) + \sum_{i=1}^{m} \frac{r_i}{1 - F(T_i)} (\frac{\partial[1 - F(T_i)]}{\partial \hat{\beta}}) = 0 \qquad (3.4)$$

Where

$$\frac{\partial F(T_i)}{\partial \hat{\beta}} = -\hat{\theta}(P_i)^{\hat{\theta}-1}(1-P_i)\ln(1-P_i)\ln(T_i/\hat{\alpha})$$

and

$$\frac{\partial\left[1-F(T_j)\right]}{\partial\hat{\beta}}=-\frac{\partial F(T_j)}{\partial\hat{\beta}}$$

$$\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial F(T_j)}{\partial\hat{\theta}}-\frac{\partial F(T_{j-1})}{\partial\hat{\theta}}\right)+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\hat{\theta}}\right)=0 \qquad (3.5)$$

Where

$$\frac{\partial F(T_j)}{\partial\hat{\theta}}=\left(P_j\right)^{\hat{\theta}}\ln P_j$$

and

$$\frac{\partial\left[1-F(T_j)\right]}{\partial\hat{\theta}}=-\frac{\partial F(T_j)}{\partial\hat{\theta}}$$

Again, to solve the system of the non linear equations (3.3),(3.4) and (3.5), restoring to numerical techniques.

The elements of the sample information matrix, for progressively type I interval censored sample will be

$$\frac{\partial^2\ln L_1\left(X;\alpha,\beta,\theta\setminus R\right)}{\partial\alpha^2}=\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial^2 F(T_j)}{\partial\alpha^2}-\frac{\partial^2 F(T_{j-1})}{\partial\alpha^2}\right)$$
$$-\frac{k_j}{\left[F(T_j)-F(T_{j-1})\right]^2}\left(\frac{\partial F(T_j)}{\partial\alpha}-\frac{\partial F(T_{j-1})}{\partial\alpha}\right)^2+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial^2\left[1-F(T_j)\right]}{\partial\alpha^2}\right)$$
$$-\sum_{j=1}^{m}\frac{r_j}{\left[1-F(T_j)\right]^2}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\alpha}\right)^2$$

Where

$$\frac{\partial^2 F(T_j)}{\partial\alpha^2}=\theta\left(\theta-1\right)\left(P_j\right)^{\theta-2}\left[\left(1-P_j\right)\left(\beta/\alpha\right)\ln\left(1-P_j\right)\right]^2+\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\left[\left(\beta/\alpha\right)\ln\left(1-P_j\right)\right]^2$$
$$-\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\ln\left(1-P_j\right)\left(\beta/\alpha^2\right)+\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\ln\left(1-P_j\right)\left(\beta/\alpha\right)^2,$$

and

$$\frac{\partial^2\left[1-F(T_j)\right]}{\partial\alpha^2}=-\frac{\partial^2 F(T_j)}{\partial\alpha^2}$$

$$\frac{\partial^2\ln L_1\left(X;\alpha,\beta,\theta\setminus R\right)}{\partial\beta^2}=\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial^2 F(T_j)}{\partial\beta^2}-\frac{\partial^2 F(T_{j-1})}{\partial\beta^2}\right)$$
$$-\frac{k_j}{\left[F(T_j)-F(T_{j-1})\right]^2}\left(\frac{\partial F(T_j)}{\partial\beta}-\frac{\partial F(T_{j-1})}{\partial\beta}\right)^2+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial^2\left[1-F(T_j)\right]}{\partial\beta^2}\right)$$
$$-\sum_{j=1}^{m}\frac{r_j}{\left[1-F(T_j)\right]^2}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\beta}\right)^2$$

Where

$$\frac{\partial^2 F(T_j)}{\partial\beta^2}=-\theta\left(\theta-1\right)\left(P_j\right)^{\theta-2}\left[\left(1-P_j\right)\ln\left(T_j/\alpha\right)\ln\left(1-P_j\right)\right]^2-\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\left[\ln\left(T_j/\alpha\right)\ln\left(1-P_j\right)\right]^2$$
$$-\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\ln\left(1-P_j\right)\left[\ln\left(T_j/\alpha\right)\right]^2,$$

and

$$\frac{\partial^2\left[1-F(T_j)\right]}{\partial\beta^2}=-\frac{\partial^2 F(T_j)}{\partial\beta^2}$$

$$\frac{\partial^2 \ln L_1\left(X;\alpha,\beta,\theta\setminus R\right)}{\partial\theta^2}=\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial^2 F(T_j)}{\partial\theta^2}-\frac{\partial^2 F(T_{j-1})}{\partial\theta^2}\right)$$
$$-\frac{k_j}{\left[F(T_j)-F(T_{j-1})\right]^2}\left(\frac{\partial F(T_j)}{\partial\theta}-\frac{\partial F(T_{j-1})}{\partial\theta}\right)^2+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial^2\left[1-F(T_j)\right]}{\partial\theta^2}\right)$$
$$-\sum_{j=1}^{m}\frac{r_j}{\left[1-F(T_j)\right]^2}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\theta}\right)^2$$

Where
$$\frac{\partial^2 F(T_j)}{\partial\theta^2}=\left(P_j\right)^{\theta}\left[\ln P_j\right]^2$$

and

$$\frac{\partial^2\left[1-F(T_j)\right]}{\partial\theta^2}=-\frac{\partial^2 F(T_j)}{\partial\theta^2}$$

$$\frac{\partial^2 \ln L_1\left(X;\alpha,\beta,\theta\setminus R\right)}{\partial\alpha\partial\beta}=\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial^2 F(T_j)}{\partial\alpha\partial\beta}-\frac{\partial^2 F(T_{j-1})}{\partial\alpha\partial\beta}\right)$$
$$-\frac{k_j}{\left[F(T_j)-F(T_{j-1})\right]^2}\left(\frac{\partial F(T_j)}{\partial\alpha}-\frac{\partial F(T_{j-1})}{\partial\alpha}\right)\left(\frac{\partial F(T_j)}{\partial\beta}-\frac{\partial F(T_{j-1})}{\partial\beta}\right)$$
$$+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial^2\left[1-F(T_j)\right]}{\partial\alpha\partial\beta}\right)-\sum_{j=1}^{m}\frac{r_j}{\left[1-F(T_j)\right]^2}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\alpha}\right)\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\beta}\right),$$

Where
$$\frac{\partial^2 F(T_j)}{\partial\alpha\partial\beta}=\theta\left(\theta-1\right)\left(P_j\right)^{\theta-2}\left[\left(1-P_j\right)\ln\left(1-P_j\right)\right]^2\left(\beta/\alpha\right)\ln\left(T_j/\alpha\right)$$
$$+\theta\left(P_j\right)^{\theta-1}\left(\beta/\alpha\right)\left(1-P_j\right)\ln\left(T_j/\alpha\right)\left[\ln\left(1-P_j\right)\right]^2$$
$$+\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\ln\left(1-P_j\right)\left(\beta/\alpha\right)\ln\left(T_j/\alpha\right)$$
$$+\theta\left(P_j\right)^{\theta-1}\left(1-P_j\right)\ln\left(1-P_j\right)\left(1/\alpha\right),$$

and

$$\frac{\partial^2\left[1-F(T_j)\right]}{\partial\,\partial\alpha\,\beta}=-\frac{\partial^2 F(T_j)}{\partial\,\partial\alpha\,\beta}$$

$$\frac{\partial^2 \ln L_1\left(X;\alpha,\beta,\theta\setminus R\right)}{\partial\alpha\partial\;\theta}=\sum_{j=1}^{m}\frac{k_j}{F(T_j)-F(T_{j-1})}\left(\frac{\partial^2 F(T_j)}{\partial\alpha\partial\;\theta}-\frac{\partial^2 F(T_{j-1})}{\partial\alpha\partial\;\theta}\right)$$
$$-\frac{k_j}{\left[F(T_j)-F(T_{j-1})\right]^2}\left(\frac{\partial F(T_j)}{\partial\alpha}-\frac{\partial F(T_{j-1})}{\partial\alpha}\right)\left(\frac{\partial F(T_j)}{\partial\theta}-\frac{\partial F(T_{j-1})}{\partial\theta}\right)$$
$$+\sum_{j=1}^{m}\frac{r_j}{1-F(T_j)}\left(\frac{\partial^2\left[1-F(T_j)\right]}{\partial\alpha\partial\;\theta}\right)-\sum_{j=1}^{m}\frac{r_j}{\left[1-F(T_j)\right]^2}\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\alpha}\right)\left(\frac{\partial\left[1-F(T_j)\right]}{\partial\theta}\right)$$

Where

$$\frac{\partial^2 F(T_i)}{\partial\alpha\,\partial\theta} = (P_i)^{\theta-1}(\beta/\alpha)(1-P_i)\ln(1-P_i)$$
$$+\theta(P_i)^{\theta-1}(\beta/\alpha)(1-P_i)\ln(1-P_i)\ln(P_i),$$

and

$$\frac{\partial^2[1-F(T_i)]}{\partial\alpha\,\partial\theta} = -\frac{\partial^2 F(T_i)}{\partial\alpha\,\partial\theta}$$

$$\frac{\partial^2 \ln L_1(X;\alpha,\beta,\theta \setminus R)}{\partial\beta\,\partial\theta} = \sum_{i=1}^{m}\frac{k_i}{F(T_i)-F(T_{i-1})}\left(\frac{\partial^2 F(T_i)}{\partial\beta\,\partial\theta}-\frac{\partial^2 F(T_{i-1})}{\partial\beta\,\partial\theta}\right)$$
$$-\frac{k_i}{[F(T_i)-F(T_{i-1})]^2}\left(\frac{\partial F(T_i)}{\partial\beta}-\frac{\partial F(T_{i-1})}{\partial\beta}\right)\left(\frac{\partial F(T_i)}{\partial\theta}-\frac{\partial F(T_{i-1})}{\partial\theta}\right)$$
$$+\sum_{i=1}^{m}\frac{r_i}{1-F(T_i)}\left(\frac{\partial^2[1-F(T_i)]}{\partial\beta\,\partial\theta}\right)-\sum_{i=1}^{m}\frac{r_i}{[1-F(T_i)]^2}\left(\frac{\partial[1-F(T_i)]}{\partial\beta}\right)\left(\frac{\partial[1-F(T_i)]}{\partial\theta}\right)$$

Where
$$\frac{\partial^2 F(T_i)}{\partial\beta\,\partial\theta} = -(P_i)^{\theta-1}\ln(T_i/\alpha)(1-P_i)\ln(1-P_i)$$
$$-\theta(P_i)^{\theta-1}\ln(T_i/\alpha)(1-P_i)\ln(1-P_i)\ln(P_i),$$

and

$$\frac{\partial^2[1-F(T_i)]}{\partial\beta\,\partial\theta} = -\frac{\partial^2 F(T_i)}{\partial\beta\,\partial\theta}$$

Therefore the approximate sample information matrix will be

$$I_1(\hat\alpha,\hat\beta,\hat\theta) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\beta^2} & \frac{\partial^2 \ln L}{\partial\beta\partial\theta} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & \frac{\partial^2 \ln L}{\partial\beta\partial\theta} & \frac{\partial^2 \ln L}{\partial\theta^2} \end{bmatrix}_{\substack{\alpha=\hat\alpha\\ \beta=\hat\beta\\ \theta=\hat\theta}} \qquad (2.11)$$

For large n, ($n \leq 50$), matrix (2.11) is a reasonable approximation to the inverse of the Fisher information matrix.

Mle with Random Removal: Under random removal, suppose that is a random variable; the joint likelihood function of progressive interval type I censored will be

$$L(X,R;\alpha,\beta,\theta) = L_1(X;\alpha,\beta,\theta \setminus R).P(R)$$

where r_i is independent of X_i and $P(R)$ is the joint probability distribution of removals defined as

$$P(R)=\frac{(n-m)!}{\prod_{j=1}^{m} r_j!(n-m-\sum_{j=1}^{m-1} r_j)!}\pi^{\sum_{j=1}^{m-1} r_j}(1-\pi)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j} \quad (4.1)$$

since $P(R)$ does not involve the parameter θ; while $L_1(X;\ \alpha,\beta,\theta\ \backslash R)$ is the likelihood function for progressive type I interval censored with fixed removal defined in (4.1) and in particular, the likelihood function with random removal will be

$$L(X,R,\alpha,\beta,\theta)=\left\{\prod_{j=1}^{m}\left[\left[1-e^{-(T_j/\alpha)^{\gamma}}\right]^{\theta}-\left[1-e^{-(T_{j-1}/\alpha)^{\gamma}}\right]^{\theta}\right]^{k_j}\left[1-\left[1-e^{-(T_j/\alpha)^{\gamma}}\right]^{\theta}\right]^{r_j}\right\}$$
$$\left\{\frac{(n-m)!}{\prod_{j=1}^{m} r_j!(n-m-\sum_{j=1}^{m-1} r_j)!}\pi^{\sum_{j=1}^{m-1} r_j}(1-\pi)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}\right\}, \quad (4.2)$$

The maximum likelihood estimators of $\hat{\alpha},\hat{\beta}$ and $\hat{\theta}$ are found directly by maximizing the logarithm of the likelihood function in (3.2), since $P(R)$ does note involve the parameters. Therefore, the MLE $\hat{\pi}$ of π can be found by maximizing $P(R)$ directly, that is,

$$\frac{1}{\hat{\pi}}\sum_{j=1}^{m-1} r_j-\frac{1}{1-\hat{\pi}}\left((m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j\right)=0$$

therefore, the maximum likelihood estimation of parameter $\hat{\pi}$ is given by

$$\hat{\pi}=\frac{\sum_{j=1}^{m-1} r_j}{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j-1)r_j}$$

The Fisher information matrix with random removal will be

$$I(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\pi})=\begin{bmatrix} I_1(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\pi}) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_2(\hat{\pi})\end{bmatrix} \quad (4.3)$$

Where

$$I_2(\hat{\pi})=E\left(-\frac{\partial^2 \ln L(\pi)}{\partial \pi^2}\right)$$

and

$$\frac{\partial^2 \ln P(R)}{\partial \pi^2}=\frac{-1}{\pi^2}\sum_{j=1}^{m-1} r_j-\frac{1}{(1-\pi)^2}\left[(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j\right].$$

Numerical technique is needed to obtain the Fisher information matrix and the variance-covariance matrix.

Note that under fixed and random removal the estimates based on intervals with equal length when the intervals are of equal length, so that monitoring and censoring occur periodically say $Ti = i.t$.

Special Cases: Many special cases can be obtained from results derived in sections (2) and (3); this section is concerned with these results.

- Progressive Type I Interval Censored Sample for Exponential & Weibull Distribution

  If $\theta = 1$, Weibull results in the case of progressive interval type I censored may be obtained as special case from the present results by Rashwan & Darweesh[8]. For Reyleigh distribution and if $\theta = 1$, we consider the case under progressive type I interval censored when the scale parameter $\beta = 2$. When the parameters $\theta = 1$ and $\beta = 2$, results of the present section deduced to exponential under progressive type I interval censored, these results agree with those established by Aggarwala[1].
- Progressive Type I Censoring is obtained as a special case when all k_i \` s are fixed to be one.
- Type-I Interval Censoring: If $r_i = 0$ and for $i = 1,2,.....,m\text{-}1$ and $r_m = n - k$ progressive type I interval censoring results reduces to type-I interval censoring.
- Type-I Censoring: If $r_i = 0$ for $i = 1,2,.....,m\text{-}1$, $r_m = n - k$ and all k_i \` s are fixed to be one then progressive censoring Type I reduce to single censored Type I.

Expected Duration: An experimenter may be interested to know whether the test can be completed within a specified time. The information is important for an experimenter to choose an appropriate sampling; because the time required to complete an experiment has direct implication on the cost.

Following Kandell & Anderson[5], under progressive interval type I censored and grouped data; the time of removal are fixed with $T_m = T$ being the time of experiment termination and r_m being the number of surviving units at that time. The expected duration with fixed removal will derived as follows

Length of Test	Probability of Failure	Probability of Removal
t_1	$P_1^{\,n}$	$[1- F(t_1)]^{ri}$
t_2	$(P_1 + P_2)^{n\text{-}r1} - P_1^{\,n\text{-}r1}$	$[1- F(t_2)]^{r2}$
•	•	•
•	•	•
•	•	•
t_i	$(P_1 +\dots + P_i)^{n\text{-}r1\dots ri\text{-}1} - (P_1 +\dots + P_{i-1})^{n\text{-}r1\dots ri\text{-}1}$	$[1- F(t_i)]^{ri}$
•		
•		
•		
t_{m-1}	$(P_1 +\dots + P_{m-1})^{n\text{-}r1\dots ri\text{-}2} - (P_1 +\dots + P_{m-2})^{n\text{-}r1\dots rm\text{-}1}$	$[1- F(t_{m-1})]^{rm\text{-}i}$
T	$1 - (P_1 +\dots + P_i)^{n\text{-}r1\dots ri\text{-}1}$	$[1- F(t_m)]^{rm}$

$$E(T/R)=\sum_{i=1}^{m-1} t_i\left[\left(P_1+P_2+\dots P_i\right)^{n-(r_1+\dots+r_{i-1})}-\left(P_1+P_2+\dots P_{i-1}\right)^{n-(r_1+\dots+r_{i-1})}\right]\left[1-F(t_i)\right]^{r_i}$$
$$+T\left[1-\left(P_1+P_2+\dots P_{m-1}\right)^{n-(r_1+\dots+r_{m-1})}\right]\left[1-F(T)\right]^{r_m}$$

Where

$$P_i=\left[\left(1-e^{-(t_i/\alpha)^{\beta}}\right)^{\theta}-\left(1-e^{-(t_{i-1}/\alpha)^{\beta}}\right)^{\theta}\right]$$

It is follows that the expected duration under progressive interval type I is given by

$$E\left[T/R\right]=T\left[1-(1-e^{-(T/\alpha)^{\beta}})^{\theta}\right]^{r_m}-\sum_{i=1}^{m-1}\Big\{t_{i+1}(1-e^{-(t_i/\alpha)^{\beta}})^{\theta[n-(r_1+\dots+r_i)]}\left[1-(1-e^{-(t_i/\alpha)^{\beta}})^{\theta}\right]^{r_{i+1}}$$
$$-t_i(1-e^{-(t_i/\alpha)^{\beta}})^{\theta[n-(r_1+\dots+r_{i-1})]}\left[1-(1-e^{-(t_i/\alpha)^{\beta}})^{\theta}\right]^{r_i}\Big\}. \qquad (6.1)$$

For equal spacing, since $t_i = ih$, for $i = 1,2,\dots m$; expected duration under this condition will be

$$E\left[T/R\right]=mh\left[1-(1-e^{-(mh/\alpha)^{\beta}})^{\theta}\right]^{r_m}-h\sum_{i=1}^{m-1}\Big\{(i+1)(1-e^{-(ih/\alpha)^{\beta}})^{\theta[n-(r_1+\dots+r_i)]}.$$
$$.\left[1-(1-e^{-(ih/\alpha)^{\beta}})^{\theta}\right]^{r_{i+1}}-i.(1-e^{-(ih/\alpha)^{\beta}})^{\theta[n-(r_1+\dots+r_{i-1})]}\left[1-(1-e^{-(ih/\alpha)^{\beta}})^{\theta}\right]^{r_i}\Big\}. \qquad (6.2)$$

For r_i = 0 for all i; expected duration from exponentiated Weibull distribution under grouped data is given by

$$E\left[T/R\right] = mh - h\sum_{i=1}^{m-1}\left\{(1-e^{-(ih/\alpha)^{\beta}})^{n\theta}\right\}. \qquad (6.3)$$

Expected duration from Weibull distribution under grouped data can be found by substituting θ = 1 in (6.3) as follow

$$E\left[T/R\right] = mh - h\sum_{i=1}^{m-1}\left\{(1-e^{-(ih/\alpha)^{\beta}})^{n}\right\}. \qquad (6.4)$$

By expanding the term in brackets on the right of (6.4), interchanging summations and summing over the subscript i , then

$$E(T/R) = mh - h\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}\frac{e^{-(jh/\alpha)^{\beta}} - e^{-(jT/\alpha)^{\beta}}}{1-e^{-(jh/\alpha)^{\beta}}}$$

Using of L'Hopital's rule, we have

$$\lim_{h\to 0}\frac{h\left[e^{-(jh/\alpha)^{\beta}} - e^{-(jT/\alpha)^{\beta}}\right]}{1-e^{-(jh/\alpha)^{\beta}}} = \frac{\alpha\beta}{j}\left[1-e^{-(jT/\alpha)^{\beta}}\right]$$

hence

$$\lim_{h\to 0}E(T/R) = \alpha\beta\sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}\binom{n}{j}\left[1-e^{-(jT/\alpha)^{\beta}}\right] \qquad (6.5)$$

Also, if β = 1; results for expected duration derived by Kandell & Anderson[5] can be obtained as special case from equation (6.5).

By putting r_i = 1 for i = 1,2,....,m-1 and taking r_m = n-k; expected duration under progressive interval censored type I in (6.2) deduced to expected duration under interval censored type I as follow

$$E\left[T/R\right] = mh\left[1-(1-e^{-(mh/\alpha)^{\beta}})^{\theta}\right]^{n-k}\left[1-(1-e^{-[(m-1)h/\alpha]^{\beta}})^{n}\right] + (m-1)h(1-e^{-[(m-1)h/\alpha]^{\beta}})^{n}$$
$$- h\sum_{i=1}^{m-2}\left\{(1-e^{-(ih/\alpha)^{\beta}})^{n}\right\}. \qquad (6.6)$$

The time of complete sampling with n test units is given by $E(X_{(n)})$, then; the expected value of the largest order statistics $(X_{(n)})$ from exponentiated Weibull distribution will be

$$E(X_{(n)}) = n\theta\alpha\Gamma\left(\frac{1}{\beta}+1\right)\sum_{j=0}^{n\theta-1}\binom{n\theta-1}{j}(-1)^{j}\left[\frac{1}{j+1}\right]^{\left(\frac{1}{\beta}+1\right)} \qquad (6.7)$$

Under type I progressively interval censoring with random removals, the R terms are random; so the expected time to complete an experiment under this type is given by taking the expectation of equation (6.1) with respect to the R terms. The calculation is rather cumbersome, but it is given by

$$\mathrm{E}(T) = \mathrm{E}_{R}\left[\mathrm{E}\left[T/R\right]\right]$$
$$= \sum_{r_1=0}^{g(r_1)}\sum_{r_2=0}^{g(r_2)}\cdots\sum_{r_{m-1}=0}^{g(r_{m-1})}P(R)\mathrm{E}\left[T/R\right] \qquad (6.8)$$

where $g(r_i)$ = n-m-r_i-...-r_i and $P(R)$ is given in equation (4.1). thus, equation (6.1) gives an expression to compute the expected time for given values of m and n

The ratio of the expected time under different schemes to the expected time under complete sampling namely; ratio of expected experiment times (REET).

$$REET = \frac{Expected\ \ experiment\ \ time\ \ Under\ \ different\ \ schemes}{Expected\ \ experiment\ \ time\ \ Under\ \ Complete\ \ Sample} \qquad (6.9)$$

Note that the REET does not depend on the scale parameter . Suppose that an experimenter wants to observe the failure of at least k complete failures when the test is anticipated to be conducted under different schemes. Then, the REET provides important information in determining whether the experiment time can be shortened significantly if a much larger sample of n test units is used and the test is stopped once k failures are observed.

A Numerical Illustration: There are no explicit forms for obtaining estimators for the exponentiated Weibull distribution under progressively type I interval censored samples based on random removals. Therefore, numerical solution and computer facilities are needed.

Using "MATHCAD" (2001), a sample size 50 was generated from the exponentiated Weibull, with parameters α = 400, β = .33 and θ = 2 based on progressive type I interval censoring with random removal. The number of surviving items r_i is assumed to follow a binomial distribution with parameters n π. Where the removal probability π takes the value 0.5 and the sample size of a binomial distribution n takes value 4, where r_1 = 0, r_2 =1, r_3 = 2, r_4 = 1 and the value of is equal to all remaining units at time T_5 . The results are:

0.3575	4145.30708	3351.54183	1247.90791	553.189694
0.3643	5461.59924	4042.50597	1331.44001	578.206387
2109.9	709.993006	7.81963961	6176.65796	74.8782122
2319.8	824.904223	18.097692	6301.14553	151.323126
2419	838.436423	29.5466397	7569.76306	163.906121
3013.3	898.681567	41.1364375	8556.98742	341.889786
3186.3	940.080164	44.4230497	22913.7974	471.388146
44.545	28434.094	3200.5292	1140.16731	475.522593
46.958	57635.2122	1352.72559	592.897951	56.3712566
47.995	112804.813	1431.66549	680.986265	58.2438191

To check adequacy of these models to these generated data, and using Chi-square goodness of fit test is carried out, we conclude that the models provides a good fit to the present data at 5% level of significance.

Suppose that progressive interval censored type I form the exponentiated Weibull family with binomial random removal occurs at five stages $m = 5$. Assume that at time T_1 = .364, none unit selected at random from the survivors, were censored (i.e. removed from the test). At T_2 = 29.547, one additional randomly selected survivor was removed. Two additional randomly select survivor was removed at T_3 = 44.545. At time T_4 = 58.244, Another one unit selected at random from the survivors, and the test was terminated at T_5 = 151.323 with thirty-two survivor. Therefore, using simulated data, we have the following

- None unit selected at random from the survivors at time T_1 = .364 and the units are observed at this time are 0.357; 0.364.
- At T_2 = 29.547, the censored item is 898.682 and the units known to have failed in the interval [T_1; T_2] are 7.82, 18.098, 29.547.
- The units removed from the test at T_3 = 44.545 are 838.436, and the units are observed form the test at the next interval T_2 ,T_3 are 41.136, 44.423, 44.54.
- The unit removed from the test at T_4 = 58.244 is 163.906 and the units are observed in the interval T_3 , T_4 are 46.958, 47.995, 56.371, 58.244.
- The remaining survivors until the time T_5 = 151.323 are

341.89	5461.59	2319.75	824.90	341.89	5461.599	2319.750	824.90
471.39	6176.65	2419.04	940.08	471.39	6176.65	2419.04	940.08
475.52	6301.14	3013.32	1140.16	475.52	6301.14	3013.32	1140.16
553.19	7569.76	3186.31	1247.90	553.19	7569.76.	3186.31	1247.90

and the units known to have failed in the last interval [T_4; T_5] are 74.878, 151.323. In summarizing data, we record:
n = 50

T_1 = .364	T_2 = 29.547	T_3 = 44.545	T_4 = 58.244	T_5 = 151.323
k_1 = 2	k_2 = 3	k_3 = 3	k_4 = 4	k_5 = 2
r_1 = 0	r_2 = 1	r_3 = 2	r_4 = 1	r_5 = 32

Using the mathematical computing package "MATHCAD" (2001) and using equations in section (4.2.1), maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ are calculated, i.e., we have

$$\hat{\alpha} = 463.267, \hat{\beta} = 0.27 \text{ and } \hat{\theta} = 1.9$$

Again, using a computing package "MATHCAD", the approximate variances and covariance of the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ were calculated as described in section (4.2.1) and are given as

$Var(\hat{\alpha}) = 3.281\times10^{4}$ $\quad Var(\hat{\beta}) = 0.098$ $\quad Var(\hat{\theta}) = 8.03$

$Cov(\hat{\alpha},\hat{\beta}) = 528.103$ $\quad Cov(\hat{\alpha},\hat{\theta}) = -5.041\times10^{-3}$ $\quad Cov(\hat{\beta},\hat{\theta}) = -0.863$

Under fixed removal; maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ form the exponentiated Weibull family under progressive interval censored type I are obtained as

$$\hat{\alpha} = 413.439, \hat{\beta} = 0.327 \text{ and } \hat{\theta} = 2.128$$

Under fixed removal; maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ form the exponentiated Weibull family under progressive interval censored type I are obtained as

$Var(\hat{\alpha}) = 6.958\times10^{4}$ $\quad Var(\hat{\beta}) = 0.293$ $\quad Var(\hat{\theta}) = 32.018$

$Cov(\hat{\alpha},\hat{\beta}) = 1.474\times10^{3}$ $\quad Cov(\hat{\alpha},\hat{\theta}) = -1.501\times10^{4}$ $\quad Cov(\hat{\beta},\hat{\theta}) = -3.11$

The value of $\hat{\alpha}$ in the case of progressive type I interval with binomial removals are greater than the corresponding in the fixed case, whilst the values of $\hat{\beta}$ and $\hat{\theta}$ in the case of progressive type I interval with binomial removals are less than in the fixed case.

As a special case, for progressive type I censored data with random removal; suppose that time T_1 = .364, $T2$ = 29.547, T_3 = 44.545, T_4 = 58.244 and T_5 = 151.323. Survivor units 0, 1, 2, 1 and 32 are removed from the test, respectively. Thus, we have n = 50, k = 14 failed units. We have the following realizations

T_1 = .364,	$T2$ = 29.547,	T_3 = 44.545,	T_4 = 58.244 and	T_5 = 151.323
r_1 = 0	r_2 = 1	r_3 = 2	r_4 = 1	r_5 = 32

The estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ are obtained as

$$\hat{\alpha} = 409.088, \hat{\beta} = 0.37 \text{ and } \hat{\theta} = 1.765$$

$Var(\hat{\alpha}) = 3.989\times10^{4}$ $\quad Var(\hat{\beta}) = 0.469$ $\quad Var(\hat{\theta}) = 19.42$

$Cov(\hat{\alpha},\hat{\beta}) = 1.333\times10^{3}$ $\quad Cov(\hat{\alpha},\hat{\theta}) = -8.738\times10^{3}$ $\quad Cov(\hat{\beta},\hat{\theta}) = -2.991$

For progressive type I censored data with fixed removal; the estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ are obtained as

$$\hat{\alpha} = 368.21, \hat{\beta} = 0.379 \text{ and } \hat{\theta} = 2.991$$

with following

$Var(\hat{\alpha}) = 2.999 \times 10^4$ $\quad Var(\hat{\beta}) = 0.299$ $\quad Var(\hat{\theta}) = 23.159$

$Cov(\hat{\alpha}, \hat{\beta}) = 981.171$ $\quad Cov(\hat{\alpha}, \hat{\theta}) = -8.396 \times 10^3$ $\quad Cov(\hat{\beta}, \hat{\theta}) = -2.674$

The value of $\hat{\alpha}$ in the case of progressive type I interval with binomial removals are greater than the corresponding in the fixed case, whilst the value of $\hat{\theta}$ in the case of progressive type I interval with binomial removals are less than in the fixed case, but $\hat{\beta}$ is equal in the two cases.

For type I interval data ($r_1 = r_2 = r_3 = r_4 = 0$), we have the following realizations

$T_1 = .364$	$T_2 = 29.547$	$T_3 = 44.545$	$T_4 = 58.244$	$T_5 = 151.323$
$k_1 = 2$	$k_2 = 3$	$k_3 = 3$	$k_4 = 4$	$k_5 = 2$
$r_1 = 0$	$r_2 = 0$	$r_3 = 0$	$r_4 = 0$	$n - r = 36$

The maximum likelihood estimates $\hat{\alpha}, \hat{\beta} \text{ and } \hat{\theta}$ for unknown parameters α, β and θ are obtained as

$$\hat{\alpha} = 600.42, \hat{\beta} = 0.267 \text{ and } \hat{\theta} = 1.829$$

with following

$Var(\hat{\alpha}) = 6.365 \times 10^4$ $\quad Var(\hat{\beta}) = 0.11$ $\quad Var(\hat{\theta}) = 8.514$

$Cov(\hat{\alpha}, \hat{\beta}) = 780.747 \times 10^3$ $\quad Cov(\hat{\alpha}, \hat{\theta}) = -7.231 \times 10^3$ $\quad Cov(\hat{\beta}, \hat{\theta}) = -0.942$

For type I censoring, let $T = 151.323$, thus, we have $k = 24$ failure, and $n - k$ survivor to be removed from the test. Using these data, we have

$$\hat{\alpha} = 386.978, \hat{\beta} = 0.334 \text{ and } \hat{\theta} = 1.934$$

the approximate variances and covariance of the maximum likelihood estimates $\hat{\alpha}, \hat{\beta} \text{ and } \hat{\theta}$ are obtained as

$Var(\hat{\alpha}) = 5.88 \times 10^4$ $\quad Var(\hat{\beta}) = 0.457$ $\quad Var(\hat{\theta}) = 29.784$

$Cov(\hat{\alpha}, \hat{\beta}) = 1.61 \times 10^3$ $\quad Cov(\hat{\alpha}, \hat{\theta}) = -1.317 \times 10^4$ $\quad Cov(\hat{\beta}, \hat{\theta}) = -3.662$

We have computed the expected duration for $m = 5$ such that $n > m$; the duration of a progressive interval type I with random removal $E(T) = 0.0128$ is shorter than progressive interval type I without removal $E(T/R) = 0.0128$. Note that, the removal probability π has an important impact on the estimation of the parameters; also, a large removal probability π means more withdrawals occurred in the process, so variance, covariance and expected decrease. Also, expected duration under interval type I censored, censored type I and complete sample are $E(T) = 0.0131$, $E(T) = 3.458$ and $E(X_{(k)}) = 12.806$ respectively.

Using ratio of expected experiment times (REET) in (6.9), progressive interval type I censored with fixed removal, progressive interval type I censored with random removal, interval type I censored and type I censored 0.003, 9.9×10^{-4}, 0.001 and 0.27 respectively. Note that; the values of the REET of different schemes and complete sampling plan decrease as n increases; also, the REET does not depend on the scale parameter .

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