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On estimation of the exponentiated Pareto distribution under different sample schemes

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ABSTRACT

Bayes and classical estimators have been obtained for a two-parameter exponentiated Pareto distribution for when samples are available from complete, type I and type II censoring schemes. Bayes estimators have been developed under a squared error loss function as well as under a LINEX loss function using priors of non-informative type for the parameters. It has been seen that the estimators obtained are not available in nice closed forms, although they can be easily evaluated for a given sample by using suitable numerical methods. The performances of the proposed estimators have been compared on the basis of their simulated risks obtained under squared error as well as under LINEX loss functions.

1. Introduction

The exponentiated Pareto distribution with a cumulative distribution function is expressed as

$$F(x;\alpha,\theta)=\left[1-(1+x)^{-\alpha}\right]^{\theta}\quad x>0,\alpha>0,\theta>0. \tag{1.1}$$

and was introduced by Gupta et al. [5] as a lifetime model. The probability density function with two shape parameters α and θ is given by

$$f(x;\alpha,\theta)=\alpha\theta\left[1-(1+x)^{-\alpha}\right]^{\theta-1}(1+x)^{-(\alpha+1)}\quad x>0,\alpha>0,\theta>0. \tag{1.2}$$

When $\theta=1$, the above distribution corresponds to the standard Pareto distribution of the second kind [see [6]].

The procedure of estimation for the exponentiated Pareto distribution under censoring seems to be untouched and, therefore, we are interested in developing a procedure of estimation for the exponentiated Pareto distribution for a censored sample case [see [8]].

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On another important issue, it is to be noted that the inferential procedures for lifetime models are often developed using squared error loss functions. No doubt the use of the squared error loss function is well justified when the loss is symmetric in nature. Its use is also very popular, perhaps, because of its mathematical simplicity. But in life testing and reliability problems, the losses are not always symmetric in nature and hence the use of a squared error loss function is forbidden and unacceptable in many situations. Inappropriateness of a squared error loss function has also been pointed out by different authors. Ferguson [4], Zellner and Geisel [14], Aitchison and Dunsmore [1], and Varian [12] introduced the LINEX loss function which is a simple generalization of the squared error loss function and can be used in almost every situation. The squared error loss function can also be considered as a particular case of the LINEX loss function [see [13,9,7], etc.]. Gupta et al. [5] showed that the exponentiated Pareto distribution can be used quite effectively in analyzing many lifetime data. The exponentiated Pareto distribution can have decreasing and upside-down bathtub shaped failure rates depending on the shape parameter θ. Shawky and Abu-Zinadah [10] studied how the different estimators of the unknown parameters of an exponentiated Pareto distribution can behave for different sample sizes and for different parameter values. Finally, Singh et al. [11] studied the exponentiated Weibull family on the basis of a type II censored scheme. Here, we mainly compare the maximum likelihood estimators with the other estimators, such as those of the method of moment estimators, estimators based on percentiles, least squares estimators, weighted least squares estimators and the estimators based on linear combinations of order statistics, mainly with respect to their biases and root mean squared errors, using extensive simulation techniques.

Recently, it has been very important to introduce and study point estimators for both of the shape parameters of the exponentiated Pareto distribution, under complete and censored type I and censored type II samples. The organization of the paper is as follows: In Section 2 maximum likelihood estimators are discussed disregarding the sample type, and the Fisher information matrix will be obtained. Bayes estimators have been developed under a squared error loss function as well as under the LINEX loss function using priors of non-informative type for the parameters in Section 3. Finally, in Section 4 an example will be discussed to illustrate the application of the results.

2. Maximum likelihood estimators

In a typical life test, n specimens are placed under observation and as each failure occurs the time is noted. Finally at some pre-determined fixed time, T, or after a pre-determined fixed number of sample specimens fail, r, the test is terminated. In both of these cases the data collected consist of observations $x_{(1)}, x_{(2)}, \ldots, x_{(r)}$ plus the information that $(n-r)$ specimens survived beyond the time of termination, T in the former case and $x_{(r)}$ in the latter. When T is fixed and r is thus a random variable, censoring is said to be of single censored type I; when r is fixed and the time of termination T is a random variable, censoring is said to be of single censored type II. In both type I and type II censoring, Cohen [2] gave the likelihood function as

$$L(X;\alpha,\theta) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_{(i)};\alpha,\theta)\,[1-F(x_0;\alpha,\theta)]^{n-r} \tag{2.1}$$

where $f(x;\alpha,\theta)$ and $F(x;\alpha,\theta)$ are the density and distribution functions respectively, and for type I the time of termination is at $x_0 = T$ and for type II at $x_0 = x_{(r)}$. If $r = n$, then Eq. (2.1) reduces to complete samples. By taking the logarithm likelihood function with the cumulative function (1.1) and the probability density function (1.2) based on Eq. (2.1), we obtain

$$\log L(X;\alpha,\theta) = \log C + r\log\alpha + r\log\theta + (\theta-1)\sum_{i=1}^{r}\log\left[1-(1+x_{(i)})^{-\alpha}\right] - (\alpha+1)\sum_{i=1}^{r}\log(1+x_{(i)}) + (n-r)\log[1-F(x_0;\alpha,\theta)] \tag{2.2}$$

where $C = n!/(n-r)!$.

Thus, the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\theta}$ can be obtained by differentiating (2.2) with respect to α and θ and equating to zero; that is, by simultaneously solving the estimating equations,

$$\frac{r}{\alpha} + (\theta - 1) \sum_{i=1}^{r} \frac{(1 + x_{(i)})^{-\alpha} \log(1 + x_{(i)})}{1 - (1 + x_{(i)})^{-\alpha}} - \sum_{i=1}^{r} \log(1 + x_{(i)}) - (n - r) \frac{\partial/\partial\alpha.F(x_0; \alpha, \theta)}{1 - F(x_0; \alpha, \theta)} = 0 \tag{2.3}$$

$$\frac{r}{\theta} + \sum_{i=1}^{r} \log[1 - (1 + x_{(i)})^{-\alpha}] - (n - r) \frac{\partial/\partial\theta.F(x_0; \alpha, \theta)}{1 - F(x_0; \alpha, \theta)} = 0 \tag{2.4}$$

and therefore

$$\hat{\alpha} = \frac{r.w}{w \sum_{i=1}^{r} \log(1 + x_{(i)}) + (n - r)\partial/\partial\alpha.F(x_0; \alpha, \theta) - w.(\theta - 1) \sum_{i=1}^{r} \left(\frac{(1+x_{(i)})^{-\alpha} \log(1+x_{(i)})}{1-(1+x_{(i)})^{-\alpha}} \right)} \tag{2.5}$$

and

$$\hat{\theta} = \frac{-r.w}{w \sum_{i=1}^{r} \log[1 - (1 + x_{(i)})^{-\hat{\alpha}}] - (n - r)\partial/\partial\theta F(x_0; \hat{\alpha}, \hat{\theta})} \tag{2.6}$$

where

$$w = [1 - F(x_0; \hat{\alpha}, \hat{\theta})],$$

$$\frac{\partial}{\partial\alpha} F(x_0; \alpha, \theta) = \theta \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta-1} (1 + x_0)^{-\alpha} \log(1 + x_0),$$

and

$$\frac{\partial}{\partial\theta} F(x_0; \alpha, \theta) = \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta} \log\left[1 - (1 + x_0)^{-\alpha}\right].$$

Note that if $r = n$, the normal equations in (2.3) and (2.4) will reduce to the normal equations from the complete sample in [10].

Again, to solve the system of non-linear equations (2.5) and (2.6), we must resort to numerical techniques and mathematical packages.

The asymptotic variance–covariance matrix of the estimators of the parameters is obtained by inverting the Fisher information matrix in which elements are negatives of expected values of the second partial derivatives of the logarithm of the likelihood function. The elements of the sample information matrix for the censored schemes sample will be

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2} \log L(X; \alpha, \theta) &= \frac{-r}{\alpha^2} - (\theta - 1) \sum_{i=1}^{r} \frac{(1 + x_{(i)})^{-\alpha} [\log(1 + x_{(i)})]^2}{[1 - (1 + x_{(i)})^{-\alpha}]^2} \\ &\quad - (n - r) \frac{\partial^2/\partial\alpha^2 F(x_0; \alpha, \theta)[1 - F(x_0; \alpha, \theta)] + [\partial/\partial\alpha F(x_0; \alpha, \theta)]^2}{[1 - F(x_0; \alpha, \theta)]^2}\end{aligned}$$

$$\frac{\partial^2}{\partial\theta^2} \log L(X; \alpha, \theta) = \frac{-r}{\theta^2} - (n - r) \frac{\partial^2/\partial\theta^2 F(x_0; \alpha, \theta)[1 - F(x_0; \alpha, \theta)] + [\partial/\partial\theta F(x_0; \alpha, \theta)]^2}{[1 - F(x_0; \alpha, \theta)]^2}$$

and

$$\frac{\partial^2}{\partial\alpha\partial\theta} \log L(X; \alpha, \theta) = \sum_{i=1}^{r} \frac{(1 + x_{(i)})^{-\alpha} \log(1 + x_{(i)})}{1 - (1 + x_{(i)})^{-\alpha}} - (n - r) \frac{\partial^2/\partial\alpha\partial\theta F(x_0; \alpha, \theta)[1 - F(x_0; \alpha, \theta)] + \partial/\partial\alpha F(x_0; \alpha, \theta)\partial/\partial\theta F(x_0; \alpha, \theta)}{[1 - F(x_0; \alpha, \theta)]^2}$$

where

$$\frac{\partial^2}{\partial\alpha^2}F(x_0;\alpha,\theta) = \theta(\theta-1)\left[1-(1+x_0)^{-\alpha}\right]^{\theta-1}\left[(1+x_0)^{-\alpha}\log(1+x_0)\right]^2 - \theta\left[1-(1+x_0)^{-\alpha}\right]^{\theta-1}(1+x_0)^{-\alpha}\left[\log(1+x_0)\right]^2,$$

$$\frac{\partial^2}{\partial\theta^2}F(x_0;\alpha,\theta) = \left[1-(1+x_0)^{-\alpha}\right]^{\theta}\log\left[1-(1+x_0)^{-\alpha}\right],$$

and

$$\frac{\partial^2}{\partial\alpha\partial\theta}F(x_0;\alpha,\theta) = \left[1-(1+x_0)^{-\alpha}\right]^{\theta-1}(1+x_0)^{-\alpha}\log(1+x_0) + \theta\left[1-(1+x_0)^{-\alpha}\right]^{\theta-1}(1+x_0)^{-\alpha}\log(1+x_0)\log\left[1-(1+x_0)^{-\alpha}\right].$$

Therefore the approximate sample information matrix will be

$$I(\hat{\alpha},\hat{\theta}) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\partial\theta} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & \frac{\partial^2 \ln L}{\partial\theta^2} \end{bmatrix}_{\substack{\alpha=\hat{\alpha} \\ \theta=\hat{\theta}}}. \tag{2.7}$$

[see [3]]. For large n $(n \geq 50)$, matrix (2.7) is a reasonable approximation to the inverse of the Fisher information matrix. Note that closed form expressions for the expected values of these second-order partial derivatives are not readily available. These terms can be evaluated by using numerical methods. Furthermore, define $V = \lim_{n\to\infty} nI_1^{-1}(\hat{\alpha},\hat{\beta},\hat{\theta})$. The joint asymptotic distribution of the maximum likelihood estimators of α and θ is multivariate normal [see [8]].

3. Bayes estimators

Consider priors of independent non-informative type for the parameters α and θ as follows:

$$g_1(\alpha) = \frac{1}{c}; \quad 0 < \alpha < c \tag{3.1}$$

and

$$g_2(\theta) = \frac{1}{\theta}; \quad \theta > 0 \tag{3.2}$$

respectively.

Combining (3.1) and (3.2) with the likelihood function (2.1) with cumulative function (1.1) and probability density function (1.2) and using the Bayes theorem, the joint posterior distribution is derived as follows:

$$\pi(\alpha,\theta) = \frac{\theta^{r-1}\alpha^r \prod_{i=1}^{r}\left[1-(1+x_{(i)})^{-\alpha}\right]^{\theta-1}(1+x_{(i)})^{-(\alpha+1)}\left\{1-\left[1-(1+x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r}}{j_1} \tag{3.3}$$

where

$$j_1 = \int_0^c\int_0^\infty \theta^{r-1}\alpha^r\prod_{i=1}^{r}\left[1-(1+x_{(i)})^{-\alpha}\right]^{\theta-1}(1+x_{(i)})^{-(\alpha+1)} \times \left\{1-\left[1-(1+x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r}\mathrm{d}\theta\,\mathrm{d}\alpha. \tag{3.4}$$

The marginal posterior of a parameter is obtained by integrating the joint posterior distribution with respect to the other parameter and hence the marginal posterior of α can be written, after simplification, as

$$\pi(\alpha/x) = \frac{\alpha^r \prod_{i=1}^{r} (1 + x_{(i)})^{-(\alpha+1)} j_2}{j_1} \tag{3.5}$$

where

$$j_2 = \int_0^\infty \theta^{r-1} \prod_{i=1}^{r} \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r} \mathrm{d}\theta.$$

Similarly, integrating the joint posterior with respect to α, the marginal posterior θ can be obtained as

$$\pi(\theta/x) = \frac{\theta^{r-1} j_3}{j_1} \tag{3.6}$$

where

$$j_3 = \int_0^c \alpha^r \prod_{i=1}^{r} (1 + x_{(i)})^{-(\alpha+1)} \prod_{i=1}^{r} \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r} \mathrm{d}\alpha.$$

The Bayes estimators for parameters α and θ of the exponentiated Pareto distribution under the squared error loss function may be defined as

$$\hat{\alpha}_{bs} = E(\alpha/x) = \int_0^c \alpha \pi(\alpha/x) \mathrm{d}\alpha$$

$$\hat{\theta}_{bs} = E(\theta/x) = \int_0^\infty \theta \pi(\theta/x) \mathrm{d}\theta$$

respectively. These estimation can be expressed as

$$\hat{\alpha}_{bs} = \frac{j_4}{j_1}$$

and

$$\hat{\theta}_{bs} = \frac{j_5}{j_1}$$

where

$$j_4 = \int_0^c \int_0^\infty \theta^{r-1} \alpha^{r+1} \prod_{i=1}^{r} \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \times \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r} \mathrm{d}\theta \mathrm{d}\alpha \tag{3.7}$$

and

$$j_5 = \int_0^c \int_0^\infty (\theta\alpha)^r \prod_{i=1}^{r} \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \times \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^{\theta}\right\}^{n-r} \mathrm{d}\theta \mathrm{d}\alpha. \tag{3.8}$$

Following Zellner [13], the Bayes estimators for the shape parameters α and θ of exponentiated Pareto distribution under the LINEX loss function are

$$\hat{\alpha}_{bl} = \frac{-1}{a} \log(E(\mathrm{e}^{-a\alpha}))$$

Table 1
Maximum likelihood estimators.

Samples	Estimators			
	$\hat{\alpha}$	$\hat{\theta}$	$MSE(\hat{\alpha})$	$MSE(\hat{\theta})$
Complete	1.824	0.447	0.031	0.00279
Type I censored	1.767	0.445	0.054	0.00306
Type II censored	1.580	0.417	0.176	0.00682

and

$$\hat{\theta}_{bl} = \frac{-1}{a} \log(E(\mathrm{e}^{-a\theta}))$$

respectively, where $E(.)$ denotes the posterior expectation. After simplification, we have

$$\hat{\alpha}_{bl} = \frac{-1}{a} \log\left(\frac{j_6}{j_1}\right) \tag{3.9}$$

and

$$\hat{\theta}_{bl} = \frac{-1}{a} \log\left(\frac{j_7}{j_1}\right) \tag{3.10}$$

where j_1 is given in (3.4),

$$j_6 = \int_0^c \int_0^\infty \mathrm{e}^{-a\alpha} \theta^{r-1} \alpha^r \prod_{i=1}^r \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \times \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^\theta\right\}^{n-r} \mathrm{d}\theta \mathrm{d}\alpha \tag{3.11}$$

and

$$j_7 = \int_0^c \int_0^\infty \mathrm{e}^{-a\theta} \theta^{r-1} \alpha^r \prod_{i=1}^r \left[1 - (1 + x_{(i)})^{-\alpha}\right]^{\theta-1} (1 + x_{(i)})^{-(\alpha+1)} \times \left\{1 - \left[1 - (1 + x_0)^{-\alpha}\right]^\theta\right\}^{n-r} \mathrm{d}\theta \mathrm{d}\alpha. \tag{3.12}$$

There are no explicit forms for obtaining estimators for the exponentiated Pareto distribution for censored scheme samples. Therefore, numerical solutions and computer facilities are needed.

4. A numerical illustration

We now illustrate the usefulness of the proposed estimators obtained in Sections 2 and 3. Using "MATHCAD" (2001), a sample of size 50 was generated from the exponentiated Pareto distribution, with parameters $\alpha = 2$ and $\theta = 0.5$.

Table 1 shows the different estimators for different sample schemes. It may be seen from Table 1 that maximum likelihood estimators are closer to the true values of α and θ than those for sampling schemes. The change in the type of scheme affects not only the maximum likelihood estimates, but also the mean square error of the schemes. It would be illogical and inappropriate to suppose that type I censored and type II censored forms perform better than the complete sample.

For Bayes estimators under different schemes the prior hyperparameters $c = 4$, 10 and 12 have been evaluated and the corresponding values are shown in Tables 2–4. These tables reveal that the Bayes estimators do not seem very sensitive to the variation of c. It is also worth mentioning that although the Bayes estimators were developed with non-informative priors, the estimated values of the Bayes estimators are not very far from the estimated values of the maximum likelihood estimators. Also, these tables show the different estimators under different sample schemes for $a = 1$, 0.01 and -1. It may be seen from these tables that Bayes estimators under the squared error loss function and

Table 2
Bayes estimators for complete samples.

Hyperparameter		$c = 4$		$c = 10$		$c = 12$	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
Squared error loss		2.273	0.53	2.273	0.53	2.273	0.53
LINEX loss function	$a = 1$	0.937	0.229	0.937	0.228	0.937	0.228
	$a = .01$	0.831	0.21	0.831	0.21	0.831	0.21
	$a = -1$	1.02	0.238	1.02	0.238	1.02	0.238

Table 3
Bayes estimators for type I samples.

Hyperparameter		$c = 4$		$c = 10$		$c = 12$	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
Squared error loss		2.221	0.431	2.272	0.434	2.272	0.434
LINEX loss function	$a = 1$	1.163	0.221	1.241	0.322	1.24	0.322
	$a = .01$	0.866	0.186	0.865	0.177	0.865	0.177
	$a = -1$	1.259	0.321	1.479	0.372	1.479	0.372

Table 4
Bayes estimators for type II samples.

Hyperparameter		$c = 4$		$c = 10$		$c = 12$	
Estimators		$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
Squared error loss		2.163	0.324	2.64	0.56	2.904	0.568
LINEX loss function	$a = 1$	0.551	0.356	0.549	0.252	0.509	0.241
	$a = .01$	0.431	0.24	0.41	0.21	0.41	0.21
	$a = -1$	0.772	0.472	0.707	0.332	0.7	0.332

the LINEX loss function are closer to the true values of α and θ as compared to maximum likelihood estimators given in Table 1. The change in the value of a affects only the Bayes estimators for LINEX loss function estimates, but on the basis of a single sample estimate, it would be illogical and inappropriate to infer that Bayes estimators under the squared error loss function and the LINEX loss function perform better than maximum likelihood estimators.

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