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Hyper Linear Exponential Distribution As a Life Distribution

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Abstract: In this paper, we introduced a new distribution called hyper linear exponential distribution and studied its properties. The maximum likelihood estimates for the unknown parameters and their variances are obtained An illustrative example is given to show the applicability of the new results.

Key words: The Linear Exponential Distribution; Maximum likelihood estimation; Failure Rate.

INTRODUCTION

The potential of the linear exponential distribution as a survival model has been demonstrated well by Broadbent[4] and Carbone *et al.*,[5]. The probability density function of linear exponential distribution with parameters λ and θ is given by the following

$$f(t;\lambda,\theta) = (\lambda+\theta t)e^{-(\lambda t+\frac{\theta t^2}{2})},$$

$$t > 0, \qquad \lambda,\theta \geq 0 \tag{1.1}$$

There is not much work done in the linear exponential distribution, most of this work based on the order statistics and record values. Some recurrence relations for both single and product moments based on the order statistics and record values have been discussed by many authors, see for example, Balakrishnan *et al*[3], Mohie El-Din *et al*[6], Abu-Youssef and Al- Ruzaiza[1], Saran and Pushkarna[8], and Ragab[7]. The corresponding results for the standard exponential and Rayleigh distributions have been deduced as the special cases which verify the results of Balakrishnan and Ahsanullah[2].

A general form for (1.1) may be obtained by adding another shape parameter say α to the linear exponential (1.1) . The probability density function of the hyper linear exponential distribution with three unknown parameters λ, θ and α is given by

$$f(t;\lambda,\theta,\alpha) = \alpha(\lambda+\theta t)(\lambda t+\frac{\theta t^2}{2})^{\alpha-1}e^{-(\lambda t+\frac{\theta t^2}{2})^{\alpha}},$$

$$t > 0, \qquad \lambda,\theta,\alpha \geq 0 \tag{1.2}$$

the cumulative function of t is given by

$$F(t;\lambda,\theta,\alpha) = (1-e^{-(\lambda t+\frac{\theta t^2}{2})^{\alpha}}) \tag{1.3}$$

and the reliability function will be

$$F(t;\lambda,\theta,\alpha) = e^{-(\lambda t+\frac{\theta t^2}{2})^{\alpha}} \tag{1.4}$$

It is clear from equation (1.2) and (1.3) that the distributions satisfy the differential equations

$$f(t;\lambda,\theta,\alpha) = \alpha(\lambda+\theta t)(\lambda t+\frac{\theta t^2}{2})^{\alpha-1}$$

$$\left[1-F(t;\lambda,\theta,\alpha)\right]$$

From (1.2), another distributions can be obtained as special cases ;that is we have the following:

1. For α, (1.2) reduces to (1.1).
2. If θ, (1.2) reduces to probability density function of the Weibull distribution.
3. For α and λ, (1.2) reduces to the Rayleigh distribution.
4. If α and θ, (1.2) reduced to the probability density function of the exponential distribution.

The new distribution (1.2) will be applicable in the fields of life testing and reliability; so we shall study its properties and estimate its parameters .The organization of the paper is as follows: Section 2 we introduce the three parameter hyper linear exponential distribution and study some of its statistical properties. The maximum likelihood estimates $\hat{\alpha},\hat{\lambda}$ and $\hat{\theta}$ and Fisher information matrix will be obtained in

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Section 3. Finally; in section 4 an example will be discussed to illustrate the application of results.

The Three Parameter:

Linear Exponential Distribution: The statistical properties play an important role in characterization of the distribution, so some statistical properties for the three parameter hyper linear exponential distribution are handled. Using (1.2), the first moment about the zero μ_1 and the second moment about the zero μ_2 of the three parameter hyper linear exponential distribution are given by:

$$\mu_1 = \frac{\alpha}{\theta}(2\theta)^{\frac{\alpha}{2}} \int_{\lambda^2/(2\theta)}^{\infty} t^{\frac{1}{2}} \left[(2\theta)^{\alpha} t - \lambda^2\right]^{\alpha-1}$$

$$e^{-\left(\frac{1}{2\theta}\right)^{\alpha}\left[(2\theta)^{\alpha} t - \lambda^2\right]^{\alpha}} dt - \frac{\lambda}{\theta} \qquad (2.1)$$

and

$$\mu_2 = \frac{\alpha}{\theta^2}(2\theta)^{\alpha} \int_{\lambda^2/(2\theta)}^{\infty} t \left[(2\theta)^{\alpha} t - \lambda^2\right]^{\alpha-1}$$

$$e^{-\left(\frac{1}{2\theta}\right)^{\alpha}\left[(2\theta)^{\alpha} t - \lambda^2\right]^{\alpha}} dt - \frac{2\lambda\mu_1}{\theta} - \frac{\lambda^2}{\theta^2} \qquad (2.2)$$

If $\alpha = 1$, the mean and variance of the linear exponential distribution can be

$$\mu_1 = \sqrt{\frac{2\pi}{\theta}} e^{\frac{\lambda^2}{2\theta}} \left[1 - \Phi\left(\frac{\lambda}{\sqrt{\theta}}\right)\right],$$

and

$$\sigma^2 = \frac{2}{\theta} - \mu_1\left(\frac{2\lambda}{\theta} + \mu_1\right),$$

respectively, where $\Phi(.)$ is cumulative distribution of the standard normal distribution.

It easy to show that the mode of the three parameter hyper linear exponential distribution can be obtained by differentiation equation (1.2), by replacing x into t_o, equating to zero and by solving it; we have

$$\alpha\theta + (\alpha-1)(\lambda + \theta t_0)^2 \left(\lambda t_0 + \frac{\theta t_0^{\,2}}{2}\right)^{-1}$$

$$-(\lambda + \theta t_0)^2 \left(\lambda t_0 + \frac{\theta t_0^{\,2}}{2}\right)^{\alpha-1} = 0 \qquad (2.3)$$

Notice that the second derivative is negative at t_o. Notice that, the mode of the linear exponential distribution may be obtained by putting $\alpha = 1$ in (2.3), that is

$$\theta - (\lambda + \theta t_0)^2 = \theta - \lambda^2 - 2\theta\lambda t_0 - \theta^2 t_0^{\,2} = 0$$

By taking $\theta = 0$ in (2.3), the mode of Weibull distribution is given by

$$t_0 = (\alpha - 1)^{\frac{1}{\alpha}} / \lambda$$

For $\alpha = 1$ and $\lambda = 0$ in (2.3), the mode of Rayleigh distribution will be

$$t_0 = (1/\theta)^{1/2}$$

It easy to show that the quantile t_q of a three parameter hyper linear exponential distribution given by

$$t_q = 2\left[-\ln(1-q)\right]^{1/\alpha} / \left[2\lambda + \theta t_q\right] \qquad (2.4)$$

For $q = 0.5$, the median of the hyper linear exponential distribution is obtained as follow

$$Median = 2\left[\ln 2\right]^{1/\alpha} / \left[2\lambda + \theta Median\right]$$

Notice that, by taking $\alpha = 1$ in (2.4), the quantile and median of the linear exponential distribution are given by

$$t_q = -2\ln(1-q) / \left[2\lambda + \theta t_q\right],$$

$$Median = 2\ln 2 / \left[2\lambda + \theta Median\right].$$

respectively. As $\theta = 0$ in (2.4), we obtained the quantile and median for the Weibull distribution as follows

$$t_q = \left[-\ln(1-q)\right]^{1/\alpha} / \lambda,$$

$$Median = \left[\ln 2\right]^{1/\alpha} / \lambda.$$

respectively. For α = 1 and λ = 1 in (2.4), the quantile and median for Rayleigh are obtained as follows

$$t_q = -2\ln(1-q)/\theta t_q \quad ,$$
$$Median = 2\ln 2/\theta Median$$

respectively. Finally, if α = 1 and θ = 1 in (2.4), the quantile and median for exponential distribution are obtained as follows

$$t_q = -\ln(1-q)/\lambda \quad ,$$
$$Median = \ln 2/\lambda \quad .$$

respectively. The failure rate *r(t)* of the hyper linear exponential distribution (α, λ, θ) can be given in the form

$$r(t) = \alpha(\lambda+\theta t)(\lambda t+\frac{\theta t^2}{2})^{\alpha-1} \tag{2.5}$$

For any λ and θ; the three parameter hyper linear exponential distribution if α <1, the hazard function decreases from ∞ to zero; if α >1, the hazard function increases from zero to ∞ and if α =1, the hazard function increasing as a linear function. Notice that, from (2.5), special cases can be obtained by putting α =1 in (2.5), the failure rate *r(t)* of the linear exponential distribution (λ, θ) is given as

$$r(t) = (\lambda+\theta t)$$

1. By taking θ = 0, the failure rate *r(t)* of the Weibull distribution (α, λ) is given by

$$r(t) = \alpha\lambda(\lambda t)^{\alpha-1}$$

2. As α = 1 and λ = 1, we obtained the failure rate *r(t)* for the Rayleigh as follows

$$r(t) = \theta t$$

3. Finally, if α = 1 and θ = 1, the failure rate *r(t)* of exponential distribution is obtained as follows

$$r(t) = \lambda$$

The Maximum Likelihood Estimation: Suppose $X_1, X_2, \ldots, X_n$ be a random sample from (1.2), then the likelihood function will be

$$L(\alpha,\lambda,\theta) = \alpha^n \prod_{i=1}^{n}(\lambda+\theta t_i)$$
$$(\lambda t_i + \frac{\theta t_i^{\,2}}{2})^{\alpha-1} e^{\sum_{i=1}^{n}(\lambda t_i + \frac{\theta t_i^{\,2}}{2})^{\alpha}} \tag{3.1}$$

The logarithm of the likelihood function given in equation (4.1) can be expressed as

$$\log L(\alpha,\lambda,\theta) = n\log\alpha + \sum_{i=1}^{n}\log(\lambda+\theta t_i)$$
$$+(\alpha-1)\sum_{i=1}^{n}\log(\lambda t_i+\frac{\theta t_i^{\,2}}{2}) - \sum_{i=1}^{n}(\lambda t_i+\frac{\theta t_i^{\,2}}{2})^{\alpha} \tag{3.2}$$

Thus, the maximum likelihood estimates $\hat{\alpha}, \hat{\lambda}$ and $\hat{\theta}$ can be obtained by differentiating (3.2) with respect to α, λ and θ; that is, by simultaneously solving the estimating equations,

$$\hat{\alpha} = \frac{1}{\ln\left[\sum_{i=1}^{n}(\hat{\lambda}t_i+\frac{\hat{\theta}t_i^{\,2}}{2})\right]}\ln\left\{\frac{n\left[1+\ln\hat{\lambda}+\ln\hat{\theta}\right]+\sum_{i=1}^{n}\ln t_i+\sum_{i=1}^{n}\ln\left(\frac{t_i^2}{2}\right)}{\ln\left[\sum_{i=1}^{n}(\hat{\lambda}t_i+\frac{\hat{\theta}t_i^{\,2}}{2})\right]}\right\} \quad , \tag{3.3}$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\hat{\alpha}-1}} \quad , \tag{3.4}$$

$$\hat{\theta} = \frac{2n}{\sum_{i=1}^{n} t_i^2 (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\hat{\alpha}-1}} \quad . \tag{3.5}$$

Again, to solve the system of the non linear equations (3.3) to (3.5), restoring to numerical techniques. The elements of the sample information matrix, for hyper linear exponential distribution will be

$$\frac{\partial \log L^2(\alpha,\lambda,\theta)}{\partial \lambda^2} = \frac{-\alpha n}{\lambda^2} - \alpha(\alpha-1)\sum_{i=1}^{n} t_i^2 (\lambda t_i + \frac{\theta t_i^2}{2})^{\alpha-2}$$

$$\frac{\partial \log L^2(\alpha,\lambda,\theta)}{\partial \theta^2} = \frac{-\alpha n}{\theta^2} - \frac{\alpha(\alpha-1)}{4}\sum_{i=1}^{n} t_i^4 (\lambda t_i + \frac{\theta t_i^2}{2})^{\alpha-2}$$

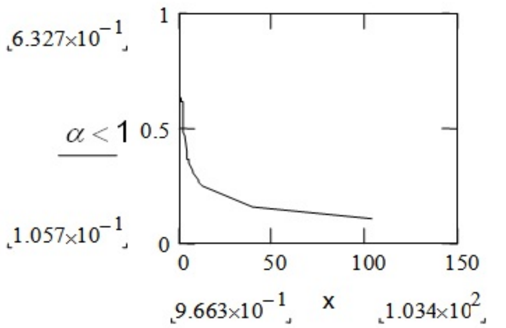
$$\frac{\partial \log L^2(\alpha,\lambda,\theta)}{\partial \alpha^2} = -\sum_{i=1}^{n} (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\alpha} \left(\ln\left\{ \sum_{i=1}^{n} (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2}) \right\} \right)^2$$

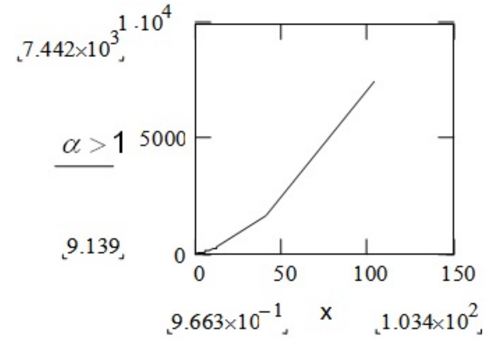
$$\frac{\partial \log L^2(\alpha,\lambda,\theta)}{\partial \theta \partial \alpha} = \frac{n}{\hat{\theta}} - \left[\frac{1}{2}\sum_{i=1}^{n} t_i^2 (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\hat{\alpha}-1} + \frac{\hat{\alpha}}{2}\sum_{i=1}^{n} t_i^2 (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\hat{\alpha}-1} \ln\left\{ \sum_{i=1}^{n} t_i^2 (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2}) \right\} \right]$$

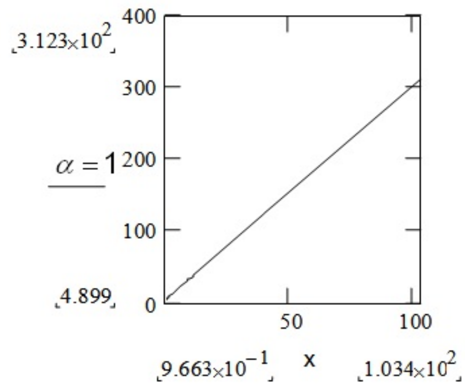
$$\frac{\partial \log L^2(\alpha,\lambda,\theta)}{\partial \lambda \partial \theta} = -\frac{\hat{\alpha}(\hat{\alpha}-1)}{2}\sum_{i=1}^{n} t_i^3 (\hat{\lambda} t_i + \frac{\hat{\theta} t_i^2}{2})^{\alpha-2}$$

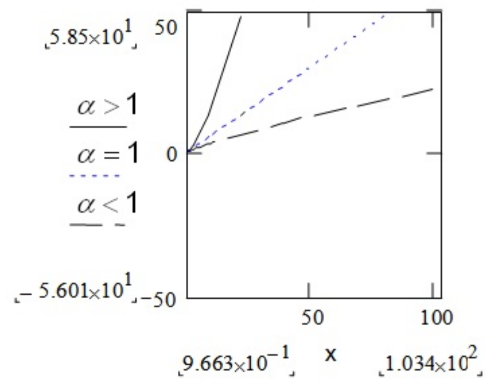
therefore the approximate sample information matrix will be

$$I_1(\hat{\alpha},\hat{\beta},\hat{\theta}) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial \theta} \\ \frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & \frac{\partial^2 \ln L}{\partial \beta \partial \theta} & \frac{\partial^2 \ln L}{\partial \theta^2} \end{bmatrix}_{\substack{\alpha=\hat{\alpha} \\ \beta=\hat{\beta} \\ \theta=\hat{\theta}}} \quad , \tag{3.6}$$









For large n, ($n \geq 50$), matrix (3.6) is a reasonable approximation to the inverse of the Fisher information matrix.

Notice that: The maximum likelihood estimates $\hat{\lambda}$ and $\hat{\theta}$ of the linear exponential distribution may be obtained by putting $\alpha = 1$ in (3.4) and (3.5).

A Numerical Illustration: There are no explicit forms for obtaining estimators for the three parameter hyper linear exponential distribution based on likelihood function. Therefore, numerical solution and computer facilities are needed. Using "MATHCAD" (2001), a sample of size 100 was generated from the hyper linear exponential distribution, with parameters $\alpha = 0.2$, $\lambda = 2$ and $\theta = 3$. To check adequacy of these models to these generated data , and using "SPSS" (11.0), Chi-square goodness of fit test is carried out, we conclude that the models provides a goodness of fit to the present data at 5% level of significance.

Test Statistics

	VAR00002
Chi-Squara[a]	2.689
df	8
Asymp. Sig.	.952

a. 0 cells (.0%) have expected frequencies less than 5. The minimum expected cell frequency is 16.4.

Using generated Data from the hyper linear exponential model, the maximum likelihood of $\hat{\alpha}$ $\hat{\lambda}$ and $\hat{\theta}$ are

$$\hat{\alpha} = 0.23 \qquad \hat{\lambda} = 1.83 \qquad \hat{\theta} = 2.67$$

Again, using a computing package "MATHCAD", the approximate variance and covariance of the maximum likelihood estimates $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ were calculated as described in section 3 and are given as

$$\text{var}(\hat{\alpha}) = 2.354 \times 10^{-2}$$

$$\text{var}(\hat{\lambda}) = 8.795 \times 10^{-3}$$

$$\text{var}(\hat{\theta}) = 2.288 \times 10^{-2}$$

$$\text{cov}(\hat{\alpha}, \hat{\lambda}) = 1.639 \times 10^{-5}$$

$$\text{cov}(\hat{\alpha}, \hat{\theta}) = 4.571 \times 10^{-5}$$

$$\text{cov}(\hat{\lambda},\hat{\theta}) = 3.783\times10^{-4}$$

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