

Estimations of the Parameters of Exponentiated Weibull Family with Type Ii Progressive Interval Censoring with Random Removals

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Abstract: This paper considers the analysis of exponentiated Weibull family distributed lifetime data observed under Type II progressive interval censoring with random removals, where the number of units removed at each failure time follows a binomial distribution. Maximum likelihood estimators of the parameters and their asymptotic variances are derived. The formula to compute the expected length of time is given. An example is discussed to illustrate the application of results under this censoring scheme.

Key words: The exponentiated Weibull family; Maximum likelihood estimation; interval censoring; Progressive Type II censoring; Random removal; Expected length of time.

INTRODUCTION

Supposed that n subjects are randomly selected at the beginning of study and the study will be terminated when there are k or more failed subjects. Let $L_i, i = 1, 2, \dots$ be the predetermined inspection times. Under a type II progressive interval censored inspection scheme, that trial is terminated after the M^{th} inspection if the total number of failed subjects is equal to or exceeds k . Suppose that at the i^{th} inspection, k_i failed subjects are observed and r_i subjects are fixed removed from the test. In other words, k_i is the number of failed subjects between any two successive inspections L_{i-1} and L_i , where $L_0 = 0$. Denoted $\zeta_j = \sum_{i=1}^j k_i$, the test is terminated when $\zeta_{M-1} < k$ and $\zeta_M \geq k$, for the predetermined integer value $k, 0 < k < n$.

Xiang & Tse^[19] point out that the $K = (k_1, k_2, \dots, k_M)$ and $R = (r_1, r_2, \dots, r_{M-1})$ where M is random and corresponds to the number of inspections before the termination of the experiment, the joint likelihood function of k_i and M , conditional on r_i , is given by

$$L(k_1, k_2, \dots, k_M, M; \theta / R) = \binom{n}{k_1} \binom{n - k_1 - r_1}{k_2} \dots \binom{n - \sum_{j=1}^{M-1} (k_j + r_j)}{k_M} \quad (1.1)$$

$$\times \prod_{i=1}^M (F(L_i) - F(L_{i-1}))^{k_i} (1 - F(L_i))^{r_i}$$

where $r_M = n - \sum_{j=1}^M k_j - \sum_{j=1}^{M-1} r_j$.

Note that k_i and M are random variables in equation (1.1), to ensure that there at least k failed subjects at the end of the study, the number of subjects removed at each inspection time, r_i , is restricted to be any integer value between 0 and $n - k - \sum_{j=1}^{i-1} r_j$, thus, r_i would not be affected by k_i for all $j = 1, 2, \dots, i$.

Xiang & Tse^[19] concluded that, the likelihood function under progressive censoring type II is obtained as a special case of equation (1.1) when all k_i 's are fixed to be 1 and $L_i = x_{(i)}$, where $x_{(i)}$ is the i^{th} ordered survival time. By all previous condition, it reduces to the type II censored if $r_i = 0$ for $i = 1, 2, \dots, m-1$ and $r_m = n - k$.

Extensive publications can be found in the literature which discuss the statistical inference for censored data under various lifetime distribution models^[1,6,9]. In particular, intensive study has been conducted for exponential lifetime data (Leslie & Eeden^[7]; Patel & Gajjar^[11]; Pettitt *et al.*^[12]; Xu & Yang.

Although progressive censoring occurs frequently in many applications, there are relatively few works on it. Some early works can be found in Cohen^[3], Mann^[8], Thomas & Wilson^[15], Viveros & Balakrishnan. Readers can refer to the book Balakrishnan & Aggarwala^[1] for more details on the methods and applications of this topic. However, all these works assumed that the number of units being removed from the test is fixed in advance. In practice, it is impossible to pre-determine the removal pattern. Thus, Yuen & Tse^[17,16] and Yang & Yuen^[20] considered the estimation problem when lifetimes collected under a Type II progressive censoring with random removals.

Model: The probability density function of the exponentiated Weibull family with two shape parameters β and θ , and scale parameter α given by

$$f(x; \alpha, \beta, \theta) = \left(\frac{\theta \cdot \beta}{\alpha}\right) \cdot \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta} \left[1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^{\theta-1} \tag{2.1}$$

Where $0 < x < \infty$, α, β and $\theta \geq 0$; the corresponding cumulative distribution function is^[10]

$$F(x; \alpha, \beta, \theta) = \left[1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^\theta \tag{2.2}$$

From equation (2.1), different special distributions can be obtained such as:

- For $\beta = 2$, the probability density function and distribution function for the exponentiated exponential distribution introduced by Gupta *et al.*^[4] will be

$$f(x; \theta, \alpha) = \left(\frac{\theta}{\alpha}\right) \cdot e^{-\left(\frac{x}{\alpha}\right)} \left(1 - e^{-\left(\frac{x}{\alpha}\right)}\right)^{\theta-1}$$

$$F(x; \alpha, \beta, \theta) = \left[1 - e^{-\left(\frac{x}{\alpha}\right)}\right]^\theta \tag{2.3}$$

respectively.

- For $\beta = 2$, the two parameter Burr type X distribution with probability density function distribution function are given by

$$f(x; \alpha, \theta) = \left(\frac{2\theta}{\alpha^2}\right) \cdot x e^{-\left(\frac{x}{\alpha}\right)^2} \left[1 - e^{-\left(\frac{x}{\alpha}\right)^2}\right]^{\theta-1}$$

$$F(x; \alpha, \beta, \theta) = \left[1 - e^{-\left(\frac{x}{\alpha}\right)^2}\right]^\theta \tag{2.4}$$

- For $\beta = 1$, the probability density function for the Weibull distribution and cumulative distribution function will be,

$$f(x; \alpha, \beta) = \left(\frac{\beta}{\alpha}\right) \cdot \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^\beta}$$

and

$$F(x; \alpha, \beta) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta} \tag{2.5}$$

respectively.

- By taking $\theta = 1$ and $\beta = 1$, the probability density function for the exponential distribution and cumulative distribution function are given by,

$$f(x; \alpha) = \left(\frac{1}{\alpha}\right) \cdot e^{-\frac{x}{\alpha}}$$

and

$$F(x; \alpha) = 1 - e^{-\frac{x}{\alpha}} \tag{2.6}$$

respectively.

- The probability density function and distribution function for the Rayleigh distribution may be obtained by putting $\theta = 1$ and $\beta = 1$, that is

$$f(x; \alpha) = \left(\frac{2}{\alpha}\right) \cdot \left(\frac{x}{\alpha}\right) \cdot e^{-\left(\frac{x}{\alpha}\right)^2}$$

and

$$F(x; \alpha) = 1 - e^{-\left(\frac{x}{\alpha}\right)^2} \tag{2.7}$$

Mle with Fixed Removal: Using the cumulative distribution function which is given by (2.2), and a progressive type II interval censored scheme suggested by Xiang & Tse^[19] in (1.1), the likelihood function and the logarithm the likelihood function as following

$$L_1(k_1, k_2, \dots, k_M, M; \theta / R) = \binom{n}{k_1} \binom{n-k_1-r_1}{k_2} \dots \binom{n-\sum_{j=1}^{M-1} (k_j+r_j)}{k_M} \tag{3.1}$$

$$\times \prod_{i=1}^M (F(L_i) - F(L_{i-1}))^{k_i} (1 - F(L_i))^{r_i}$$

$$\ln L_1(k_1, k_2, \dots, k_M, M; \theta / R) = \sum_{i=1}^M k_i \ln [F(L_i) - F(L_{i-1})] + \sum_{i=1}^M r_i \ln [1 - F(L_i)] \tag{3.2}$$

respectively.

Thus, the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ can be obtained by maximizing (3.2) with respect to α, β and θ ; that is, by simultaneously solving the estimating equations,

$$\sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial F(L_i)}{\partial \hat{\alpha}} - \frac{\partial F(L_{i-1})}{\partial \hat{\alpha}} \right) + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial [1 - F(L_i)]}{\partial \hat{\alpha}} \right) = 0 \quad (3.3)$$

where

$$\frac{\partial F(L_i)}{\partial \hat{\alpha}} = \left(\frac{\hat{\beta} \hat{\theta}}{\hat{\alpha}} \right) (P_i)^{\hat{\theta}-1} (1 - P_i) \ln(1 - P_i); P_i = 1 - e^{-(L_i / \hat{\alpha})^{\hat{\beta}}}$$

and

$$\frac{\partial [1 - F(L_i)]}{\partial \hat{\alpha}} = - \frac{\partial F(L_i)}{\partial \hat{\alpha}}$$

$$\sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial F(L_i)}{\partial \hat{\beta}} - \frac{\partial F(L_{i-1})}{\partial \hat{\beta}} \right) + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial [1 - F(L_i)]}{\partial \hat{\beta}} \right) = 0 \quad (3.4)$$

where

$$\frac{\partial F(L_i)}{\partial \hat{\beta}} = -\hat{\theta} (P_i)^{\hat{\theta}-1} (1 - P_i) \ln(1 - P_i) \ln(L_i / \hat{\alpha})$$

and

$$\frac{\partial [1 - F(L_i)]}{\partial \hat{\beta}} = - \frac{\partial F(L_i)}{\partial \hat{\beta}}$$

$$\sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial F(L_i)}{\partial \hat{\theta}} - \frac{\partial F(L_{i-1})}{\partial \hat{\theta}} \right) + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial [1 - F(L_i)]}{\partial \hat{\theta}} \right) = 0 \quad (3.5)$$

where

$$\frac{\partial F(L_i)}{\partial \hat{\theta}} = (P_i)^{\hat{\theta}} \ln P_i$$

and

$$\frac{\partial [1 - F(L_i)]}{\partial \hat{\theta}} = - \frac{\partial F(L_i)}{\partial \hat{\theta}}$$

Again, to solve the system of the non linear equations)3.3(,)3.4(and)3.5(, restoring to numerical techniques and mathematical packages.

The asymptotic variance covariance matrix of the estimators of the parameters is obtained by inverting the Fisher information matrix in which elements are negatives of expected values of the second partial derivatives of

the logarithm of the likelihood function. Denote the Fisher information matrix associated with α, β and θ by $I_1(\hat{\alpha}, \hat{\beta}, \hat{\theta})$, where

$$I_1(\hat{\alpha}, \hat{\beta}, \hat{\theta}) = E \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} & -\frac{\partial^2 \ln L}{\partial \theta^2} \end{bmatrix} \begin{matrix} \alpha = \hat{\alpha} \\ \beta = \hat{\beta} \\ \theta = \hat{\theta} \end{matrix} \quad (3.6)$$

$$\begin{aligned} \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \alpha^2} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \alpha^2} - \frac{\partial^2 F(L_{i-1})}{\partial \alpha^2} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \alpha} - \frac{\partial F(L_{i-1})}{\partial \alpha} \right)^2 + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial^2 [1 - F(L_i)]}{\partial \alpha^2} \right) \\ &- \sum_{i=1}^M \frac{r_i}{[1 - F(L_i)]^2} \left(\frac{\partial [1 - F(L_i)]}{\partial \alpha} \right)^2 \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2 F(L_i)}{\partial \alpha^2} &= \theta(\theta - 1)(P_i)^{\theta - 2} [(1 - P_i)(\beta/\alpha) \ln(1 - P_i)]^2 + \theta(P_i)^{\theta - 1} (1 - P_i) [(\beta/\alpha) \ln(1 - P_i)]^2 \\ &- \theta(P_i)^{\theta - 1} (1 - P_i) \ln(1 - P_i) (\beta/\alpha^2) + \theta(P_i)^{\theta - 1} (1 - P_i) \ln(1 - P_i) (\beta/\alpha)^2, \end{aligned}$$

and

$$\frac{\partial^2 [1 - F(L_i)]}{\partial \alpha^2} = -\frac{\partial^2 F(L_i)}{\partial \alpha^2}.$$

$$\begin{aligned} \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \beta^2} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \beta^2} - \frac{\partial^2 F(L_{i-1})}{\partial \beta^2} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \beta} - \frac{\partial F(L_{i-1})}{\partial \beta} \right)^2 + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial^2 [1 - F(L_i)]}{\partial \beta^2} \right) \\ &- \sum_{i=1}^M \frac{r_i}{[1 - F(L_i)]^2} \left(\frac{\partial [1 - F(L_i)]}{\partial \beta} \right)^2 \end{aligned}$$

where

$$\frac{\partial^2 F(L_i)}{\partial \beta^2} = -\theta(\theta-1)(P_i)^{\theta-2} [(1-P_i)\ln(L_i/\alpha)\ln(1-P_i)]^2 - \theta(P_i)^{\theta-1}(1-P_i)[\ln(L_i/\alpha)\ln(1-P_i)]^2 - \theta(P_i)^{\theta-1}(1-P_i)\ln(1-P_i)[\ln(L_i/\alpha)]^2,$$

and

$$\frac{\partial^2 [1-F(L_i)]}{\partial \beta^2} = -\frac{\partial^2 F(L_i)}{\partial \beta^2}.$$

$$\begin{aligned} \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \theta^2} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \theta^2} - \frac{\partial^2 F(L_{i-1})}{\partial \theta^2} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \theta} - \frac{\partial F(L_{i-1})}{\partial \theta} \right)^2 + \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial^2 [1 - F(L_i)]}{\partial \theta^2} \right) \\ &- \sum_{i=1}^M \frac{r_i}{[1 - F(L_i)]^2} \left(\frac{\partial [1 - F(L_i)]}{\partial \theta} \right)^2 \end{aligned}$$

where

$$\frac{\partial^2 F(L_i)}{\partial \theta^2} = (P_i)^\theta [\ln P_i]^2,$$

and

$$\frac{\partial^2 [1 - F(L_i)]}{\partial \theta^2} = -\frac{\partial^2 F(L_i)}{\partial \theta^2}.$$

$$\begin{aligned} \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \alpha \partial \beta} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \alpha \partial \beta} - \frac{\partial^2 F(L_{i-1})}{\partial \alpha \partial \beta} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \alpha} - \frac{\partial F(L_{i-1})}{\partial \alpha} \right) \left(\frac{\partial F(L_i)}{\partial \beta} - \frac{\partial F(L_{i-1})}{\partial \beta} \right) \\ &+ \sum_{i=1}^M \frac{r_i}{1 - F(L_i)} \left(\frac{\partial^2 [1 - F(L_i)]}{\partial \alpha \partial \beta} \right) - \sum_{i=1}^M \frac{r_i}{[1 - F(L_i)]^2} \left(\frac{\partial [1 - F(L_i)]}{\partial \alpha} \right) \left(\frac{\partial [1 - F(L_i)]}{\partial \beta} \right), \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2 F(L_i)}{\partial \alpha \partial \beta} &= \theta(\theta-1)(P_i)^{\theta-2} [(1-P_i)\ln(1-P_i)]^2 (\beta/\alpha)\ln(L_i/\alpha) \\ &+ \theta(P_i)^{\theta-1} (\beta/\alpha)(1-P_i)\ln(L_i/\alpha) [\ln(1-P_i)]^2 \\ &+ \theta(P_i)^{\theta-1} (1-P_i)\ln(1-P_i)(\beta/\alpha)\ln(L_i/\alpha) \\ &+ \theta(P_i)^{\theta-1} (1-P_i)\ln(1-P_i)(1/\alpha), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 [1-F(L_i)]}{\partial \alpha \partial \beta} &= -\frac{\partial^2 F(L_i)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \alpha \partial \theta} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \alpha \partial \theta} - \frac{\partial^2 F(L_{i-1})}{\partial \alpha \partial \theta} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \alpha} - \frac{\partial F(L_{i-1})}{\partial \alpha} \right) \left(\frac{\partial F(L_i)}{\partial \theta} - \frac{\partial F(L_{i-1})}{\partial \theta} \right) \\ &+ \sum_{i=1}^M \frac{r_i}{1-F(L_i)} \left(\frac{\partial^2 [1-F(L_i)]}{\partial \alpha \partial \theta} \right) - \sum_{i=1}^M \frac{r_i}{[1-F(L_i)]^2} \left(\frac{\partial [1-F(L_i)]}{\partial \alpha} \right) \left(\frac{\partial [1-F(L_i)]}{\partial \theta} \right) \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2 F(L_i)}{\partial \alpha \partial \theta} &= (P_i)^{\theta-1} (\beta/\alpha)(1-P_i)\ln(1-P_i) \\ &+ \theta(P_i)^{\theta-1} (\beta/\alpha)(1-P_i)\ln(1-P_i)\ln(P_i), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 [1-F(L_i)]}{\partial \alpha \partial \theta} &= -\frac{\partial^2 F(L_i)}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \ln L_1(k_1, k_2, \dots, k_M, M; \theta / R)}{\partial \beta \partial \theta} &= \sum_{i=1}^M \frac{k_i}{F(L_i) - F(L_{i-1})} \left(\frac{\partial^2 F(L_i)}{\partial \beta \partial \theta} - \frac{\partial^2 F(L_{i-1})}{\partial \beta \partial \theta} \right) \\ &- \frac{k_i}{[F(L_i) - F(L_{i-1})]^2} \left(\frac{\partial F(L_i)}{\partial \beta} - \frac{\partial F(L_{i-1})}{\partial \beta} \right) \left(\frac{\partial F(L_i)}{\partial \theta} - \frac{\partial F(L_{i-1})}{\partial \theta} \right) \end{aligned}$$

$$+ \sum_{i=1}^M \frac{r_i}{1-F(L_i)} \left(\frac{\partial^2 [1-F(L_i)]}{\partial \beta \partial \theta} \right) - \sum_{i=1}^M \frac{r_i}{[1-F(L_i)]^2} \left(\frac{\partial [1-F(L_i)]}{\partial \beta} \right) \left(\frac{\partial [1-F(L_i)]}{\partial \theta} \right)$$

where

$$\frac{\partial^2 F(L_i)}{\partial \beta \partial \theta} = -(P_i)^{\theta-1} \ln(L_i/\alpha)(1-P_i) \ln(1-P_i) - \theta(P_i)^{\theta-1} \ln(L_i/\alpha)(1-P_i) \ln(1-P_i) \ln(P_i),$$

and

$$\frac{\partial^2 [1-F(L_i)]}{\partial \beta \partial \theta} = - \frac{\partial^2 F(L_i)}{\partial \beta \partial \theta}$$

Not that closed form expressions of the expected values of these second order partial derivatives are not readily available. These terms can be evaluated by using numerical methods. Furthermore, define $V = \lim_{n \rightarrow \infty} nI_1^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$. The joint asymptotic distribution of the maximum likelihood estimators of α, β and θ

is multivariate normal. In particular, $\sqrt{n} \begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \\ \hat{\theta} - \theta \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, V \right)^{[6]}$.

Mle with Random Removal: Under random removal; Xiang & Tse^[19] derived the likelihood function as follows

$$L(k_1, k_2, \dots, k_M, M, R; \theta) = L_1(k_1, k_2, \dots, k_M, M; \theta / R) P(R), \tag{4.1}$$

where $L_1(k_1, k_2, \dots, k_M, M; \theta / R)$ is the likelihood function for a progressive type II interval censored with fixed removal (3.1) and $P(R)$ will be

$$\begin{aligned} P(R) &= P(R_{M-1} = r_{M-1} \setminus R_{M-2} = r_{M-2}, \dots, R_1 = r_1) P(R_{M-2} = r_{M-2} \\ &\setminus R_{M-3} = r_{M-3}, \dots, R_1 = r_1) \dots P(R_2 = r_2 \setminus R_1 = r_1) P(R_1 = r_1) \\ &= \frac{(n-k)!}{\prod_{j=1}^{M-1} r_j! (n_m - M)!} \pi^{\sum_{j=1}^{M-1} r_j} (1-\pi)^{(M-1)(n-k) - \sum_{j=1}^{M-1} (M-j)r_j} \end{aligned} \tag{4.2}$$

where $r_M = n - \sum_{j=1}^M k_j - \sum_{j=1}^{M-1} r_j$ and $P(R)$ does not involve the parameters. By assuming to follow a binomial distribution with parameter π , the probability of r_i subjects removed from the test at the i th inspection time in equation (4.2), also

$$\frac{\partial \ln P(R)}{\partial \pi} = \frac{1}{\pi} \sum_{i=1}^{M-1} r_j - \frac{1}{1-\pi} \left[(M-1)(n-k) - \sum_{j=1}^{M-1} (M-j)r_j \right], \tag{4.3}$$

Because Equation (4.3) does not depend on the parameters α, β , and θ , the maximum likelihood estimators of π is given by

$$\hat{\pi} = \frac{\sum_{j=1}^{M-1} r_j}{(M-1)(n-k) - \sum_{j=1}^{M-1} (M-j-1)r_j}$$

Note that π is only a parameter of a random removal pattern and provides no information to the survival distribution of the product. Because $P(R)$ does not depend on the parameters α, β and θ , so,

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \pi} = \frac{\partial^2 \ln L}{\partial \beta \partial \pi} = \frac{\partial^2 \ln L}{\partial \theta \partial \pi} = 0.$$

Denote the Fisher information matrix associated with α, β, θ , and π by $I(\alpha, \beta, \theta, \pi)$,

$$I(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\pi}) = \begin{bmatrix} I_1(\hat{\alpha}, \hat{\beta}, \hat{\theta}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_2(\hat{\pi}) \end{bmatrix}$$

where $I_1(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ defined in (3.6), and

$$I_2(\hat{\pi}) = E \left(- \frac{\partial^2 \ln L(\pi)}{\partial \pi^2} \right)$$

Special Cases: Many special cases can be obtained from results derived in sections (3) and (4); this section is concerned with these results.

Progressive Type II Interval Censored for Exponential & Weibull Distribution If $\theta = 1$, deduced the exponentiated Weibull random variables under progressive type II interval censored to Weibull random variable under this type, these results agree those established by Xian & Tse^[17] in section (3.3).

When the parameters $\theta = 1$ and $\beta = 1$, exponentiated Weibull random variables under progressive type II interval censored reduced to exponential distribution under the same type. For Reyleigh distribution and if $\theta = 1$, we consider the case under progressive type II interval censored when the scale parameter $\beta = 2$.

Progressive Censoring Type II is obtained as a special case when all k_i 's are fixed to be 1 and $L_i = x_{(i)}$, where $x_{(i)}$ is the i^{th} ordered survival time.

Type II Censored: By all previous condition, Progressive Type II Interval Censored reduces to the type II censored if $r_i = 0$ for $i = 1, 2, \dots, m-1$ and $r_m = n - k$.

Expected Length of Time: In practical applications, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an

appropriate sampling plan because the time required to complete a test is directly related to the cost.

Let ω denote the length of the study; the study will be terminated when k or more subjects failed. Let ζ_M be the total number of failed subjects at time L_M ; $\zeta_M = \sum_{i=1}^M k_i$ then $P_r \{ \omega = L_M / R \} = P_r \{ \zeta_M \geq k \text{ and } \zeta_M < k / R \}$.

To evaluate this probability, we consider the two cases when $M = 1$ and $M > 1$ separately. Suppose that

$$\pi_M = \frac{F(L_M) - F(L_{M-1})}{1 - F(L_{M-1})} = \frac{\left[1 - e^{-(L_M/\alpha)^\beta}\right]^\theta - \left[1 - e^{-(L_{M-1}/\alpha)^\beta}\right]^\theta}{1 - \left[1 - e^{-(L_{M-1}/\alpha)^\beta}\right]^\theta} \tag{6.1}$$

Case 1: For $M = 1$; $P_r \{ \omega = L_1 \} = P_r \{ \zeta_M \geq k \}$

$$= \sum_{i=k}^n \binom{n}{i} \cdot [F(L_1)]^i [1 - F(L_1)]^{n-i} \tag{6.2}$$

so,

$$E(L_1 / R) = \sum_{i=k}^n i \cdot \binom{n}{i} \cdot [F(L_1)]^i [1 - F(L_1)]^{n-i}$$

then, expected length of time with random removal as follow is

$$E(L_1) = E_R [E[L_1/R]] = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \dots \sum_{r_{m-1}=0}^{g(r_{m-1})} P(R) E[L_1/R] \tag{6.3}$$

Case 2: For $M > 1$

$$\begin{aligned} P_r \{ \omega = L_M / M, R \} &= P_r \{ \zeta_M \geq k \text{ and } \zeta_{M-1} < k / M, R \} \\ &= \sum_{j=0}^{k-1} P_r \{ \zeta_{M-1} = j / M, R \} \times P_r \{ \zeta_M \geq k / \zeta_{M-1} = j, M, R \} \end{aligned} \tag{6.4}$$

where

$$P_r \{ \zeta_{M-1} = j / M, R \} = \binom{n_{M-1}}{j} \cdot [F(L_{M-1})]^j [1 - F(L_{M-1})]^{n_{M-1}-j}$$

and

$$P_r \{ \zeta_M \geq k / \zeta_{M-1} = j, M, R \} = \sum_{s=k-j}^{n_M-j} \binom{n_M-j}{s} \cdot \pi_M^s (1 - \pi_M)^{n_M-j-s}$$

so,

$$E(L_M / R) = \sum_{j=0}^{k-1} j \cdot P_r \{ \zeta_{M-1} = j / M, R \} \times P_r \{ \zeta_M \geq k / \zeta_{M-1} = j, M, R \} \quad (6.5)$$

then

$$\begin{aligned} E(L_M) &= E_R [E [L_M / R]] \\ &= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \dots \sum_{r_{m-1}=0}^{g(r_{m-1})} P(R) E [L_M / R] \end{aligned}$$

where $n_i = n - \sum_{j=1}^{i-1} r_j$, $g(r_i) = n - M - r_1 - \dots - r_i$ and $P(R)$ is given in equation (4.2).

Following Yuen & Tse^[21], Expected length of time of a type II progressive censoring test without removal can be found when all k_i 's are fixed to be one and $L_i = t_{(i)}$ where $t_{(i)}$ is the i th ordered survival time will be

$$\begin{aligned} E [X_{(k)} / R] &= C(r) \sum_{l_i=0}^{r_1} \dots \sum_{l_m=0}^{r_k} (-1)^A \frac{\binom{r_1}{l_1} \dots \binom{r_k}{l_k}}{\prod_{i=1}^{k-1} h(l_i)} \int_0^\infty x f(x) F^{h(l_i)-1}(x) dx \\ &= \alpha \theta \Gamma \left(1 + \frac{1}{\beta} \right) C(r) \sum_{l_i=0}^{r_1} \dots \sum_{l_m=0}^{r_k} (-1)^A \frac{\binom{r_1}{l_1} \dots \binom{r_k}{l_k}}{\prod_{i=1}^{k-1} h(l_i)} \sum_{j=0}^{\theta h(l_i)-1} (-1)^j \binom{\theta h(l_i)-1}{j} \left(\frac{1}{j+1} \right)^{(1/\beta)+1} \end{aligned} \quad (6.6)$$

where

$$A = \sum_{i=1}^k l_i, C(r) = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \dots (n - \sum_{i=1}^{k-1} (r_i + 1)) \text{ and } h(l_i) = l_1 + \dots + l_i + i.$$

By all previous condition, expected length of time under progressive type II interval censored reduces to expected length of time of type II censoring test without removal if $r_i = 0$ for $i = 1, 2, \dots, m - 1$ and $r_m = n - k$ as follow

$$E(X_{(k)} / R) = k \alpha \theta \binom{n}{k} \Gamma \left(1 + \frac{1}{\beta} \right) \sum_{j=0}^{n-k} \sum_{l=0}^{\theta(k+j)-1} (-1)^{j+l} \binom{n-k}{j} \binom{\theta(k+j)-1}{l} \left(\frac{1}{l+1} \right)^{(1/\beta)+1} \quad (6.7)$$

If $n = k$ expected length of time of complete sampling case with n test units from exponentiated Weibull distribution can be obtained as follow

$$E(X_{(n)}) = n\theta\alpha\Gamma\left(\frac{1}{\beta} + 1\right) \sum_{l=0}^{n\theta-1} \binom{n\theta-1}{l} (-1)^l \left[\frac{1}{l+1}\right]^{\left(\frac{1}{\beta}+1\right)} \tag{6.8}$$

The time of complete sampling with n test units is given by $E(X_{(n)})$, then; the expected value of the largest order statistics $X_{(n)}$ from exponentiated Weibull distribution will be

$$E(X_{(n)}) = n\theta\alpha\Gamma\left(\frac{1}{\beta} + 1\right) \sum_{j=0}^{n\theta-1} \binom{n\theta-1}{j} (-1)^j \left[\frac{1}{j+1}\right]^{\left(\frac{1}{\beta}+1\right)} \tag{6.9}$$

The ratio of the expected time under different schemes to the expected time under complete sampling namely; ratio of expected experiment times (REET).

$$REET = \frac{\text{Expected experiment time Under different schemes}}{\text{Expected experiment time Under Complete Sample}} \tag{6.10}$$

Note that the REET does not depend on the scale parameter α . Suppose that an experimenter wants to observe the failure of at least k complete failures when the test is anticipated to be conducted under different schemes. Then, the REET provides important information in determining whether the experiment time can be shortened significantly if a much larger sample of n test units is used and the test is stopped once k failures are observed.

A Numerical Illustration: There are no explicit forms for obtaining estimators for the exponentiated Weibull distribution under progressively type II interval censored samples based on random removals. Therefore, numerical solution and computer facilities are needed.

Using “MATHCAD” (2001), a sample size 50 was generated from the exponentiated Weibull, with parameters $\alpha = 400, \beta = .33$ and $\theta = 2$ based on progressive type II interval censoring with random removal.

Suppose that $n = 50$ subjects are randomly selected at beginning of the study and the study will be terminated when there are $k = 25$ or more failed. The trial is terminated after the M th inspection if the total number of failed subjects is equal to or exceeds $k = 25$. The results are

0.3808	4058.89755	1337.95834	590.799515	57.3752252
0.3881	5334.48211	1415.31312	677.729774	59.263688
8.1018	6026.21471	2078.52523	706.33232	76.0171115
18.61	6146.56512	2283.30517	819.542732	152.654109
30.248	7371.86852	2380.14174	832.86488	165.228838
41.988	8324.08605	2959.00224	892.151797	342.373628
45.311	22093.2139	3127.3036	932.87102	470.691274
45.434	27362.5582	3141.12487	1129.45859	474.782255
47.873	55111.3181	3287.96863	1235.18208	551.576157
48.921	107214.406	3959.13494	1317.09331	576.290203

To check adequacy of these models to these generated data, and using Chi-square goodness of fit test is carried out, we conclude that the models provides a good fit to the present data at 5% level of significance. Using simulated data, we have the following:

- Number of failed subjects $k_j = 5$ between two successive inspections $L_1 = 30.248$ and $L_0 = 0$ are 0.381, 0.388, 8.102, 18.61 and 30.248. None subjects selected at random from this inspection time $r_j = 0$.

- The second inspection time $L_2 = 165.299$; Number of failed subjects $k_2 = 10$ between L_2 and L_1 are 41.988, 45.31, 45.434, 47.873, 48.92, 57.375, 59.264, 76.017, 152.654 and 165.229. The subjects removed from the test are $r_2 = 1$ is 3.959×10^3 .
- Number of failed subjects $k_3 = 5$ at third inspections $L_3 = 576.29$ are 342.374, 470.691, 474.782, 551.576 and 576.29. Two additional randomly selected from this inspection time; where $r_3 = 2$ (590.8, 3.141×10^3).
- The fourth inspection time L_4 ; Number of failed subjects k_4 between L_4 and L_3 are 677.73, 706.332 and 819.543. The subject removed from the test are $r_4 = 1$ is 7.372×10^3 .
- Number of failed subjects $k_5 = 4$ at last inspection $L_5 = 1.129 \times 10^3$ are 823.865, 892.152, 932.871 and 1.129×10^3 . The remaining survivors until the inspection $L_5 = 1.129 \times 10^3$ are

1235.2	2283.30517	4058.89755	22093.2139
1317.1	2380.14174	5334.48211	27362.5582
1338	2959.00224	6026.21471	55111.3181
1415.3	3141.12487	6146.56512	107214.406
2078.5	3287.96863	8324.08605	

In summarizing these data, we record:

$L^1 = 30.248$	$L_2 = 165.229$	$L_3 = 576.29$	$L_4 = 819.543$	$L_5 = 1.129 \times 10^3$
$k_1 = 5$	$k_2 = 10$	$k_3 = 5$	$k_4 = 3$	$k_5 = 4$
$r_1 = 0$	$r_2 = 1$	$r_3 = 2$	$r_4 = 1$	$r_5 = 19$

Using the mathematical computing package “MATHCAD” (2001) and equations in section (3), maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ are calculated, i.e., we have

$$\hat{\alpha} = 390.588, \hat{\beta} = 0.328 \text{ and } \hat{\theta} = 2.117$$

Again, using a computing package “MATHCAD”, the approximate variances and covariance of the maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ were calculated as described in section (3) and are given as

$Var(\hat{\alpha}) = 1.274 \times 10^8$	$Var(\hat{\beta}) = 4.844$	$Var(\hat{\theta}) = 617.716$
$Cov(\hat{\alpha}, \hat{\beta}) = 2.485 \times 10^4$	$Cov(\hat{\alpha}, \hat{\theta}) = -2.805 \times 10^5$	$Cov(\hat{\beta}, \hat{\theta}) = -54.722$

As a special case, progressive type II censored data (when all k_i 's are fixed to be 1 and $L_i = x_{(i)}$, where $x_{(i)}$ is the i^{th} ordered survival time), we have the following realizations:

$x_{(1)} = 0.177$	$x_{(2)} = 0.181$	$x_{(3)} = 13.279$	$x_{(4)} = 35.659$	$x_{(5)} = 37.903$
$r_1 = 0$	$r_2 = 1$	$r_3 = 2$	$r_4 = 1$	$r_5 = 41$

The estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ for unknown parameters α, β and θ are obtained as

$$\hat{\alpha} = 65.455, \hat{\beta} = 0.378 \text{ and } \hat{\theta} = 3.077$$

with following

$Var(\hat{\alpha}) = 73.03$	$Var(\hat{\beta}) = 7.839 \times 10^{-3}$	$Var(\hat{\theta}) = 0.334$
$Cov(\hat{\alpha}, \hat{\beta}) = 3.746$	$Cov(\hat{\alpha}, \hat{\theta}) = -14.512$	$Cov(\hat{\beta}, \hat{\theta}) = -0.088$

For type II censored; let $x_{(5)} = 37.903, r_1 = r_2 = r_3 = r_4 = 0$ and $r_5 = n - k$. From the realizations, we obtained the following estimates

$\hat{\alpha} = 68.711, \hat{\beta} = 0.378$ and $\hat{\theta} = 3.081$
with following

$Var(\hat{\alpha}) = 145.889$	$Var(\hat{\beta}) = 0.046$	$Var(\hat{\theta}) = 0.446$
$Cov(\hat{\alpha}, \hat{\beta}) = -5.327$	$Cov(\hat{\alpha}, \hat{\theta}) = 20.234$	$Cov(\hat{\beta}, \hat{\theta}) = -0.117$

For a given value of M , the expected length of time under progressive type II interval censored decreases as the sample size increases. The duration of a progressive type II interval censored with fixed removal [$E(L_M / R) = 7.117$] is longer than a progressive type II interval censored with random removals [$E(L_M) = 2.53$]. Also, expected length of time under progressive type II censored, type II censored and complete sample are $E(X_{(k)} / R) = 3.449, E(X_{(k)}) = 1.226$ and $E(X_{(k)}) = 12.806$ respectively.

Using ratio of expected experiment times (REET) in (6.10), progressive interval type II censored with fixed and random removal, progressive type II censored with random removal and type II censored 0.556, 0.198, 0.269 and .0957 respectively. Note that; the values of the REET of different schemes and complete sampling plan decrease as n increases; also, the REET does not depend on the scale parameter α .

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